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Mathematisch-Naturwissenschaftliche Fakultät

Amplitude Equations for the generalised Swift-Hohenberg equation with Noise

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1 Introduction

The Swift-Hohenberg equation is a model equation used to study pattern formation in driven systems. It was originally derived in [SH77] as a qualitative description of the convective instability in the Rayleigh Bernard model. Originally, it takes the form

$$\partial_t u = ru - (1+\nabla^2)^2 u - u^3 \ , \tag{1.1}$$

where $r \in \mathbb{R}$ is the bifurcation parameter. At $r = 0$ is the change of stability that corresponds to the convective instability. A variant is the so called generalised Swift-Hohenberg model with quadratic and cubic nonlinearity:

$$\partial_t u = ru - (1+\nabla^2)^2 u + \vartheta u^2 - u^3, \tag{1.2}$$

where $\vartheta > 0$ is an additional parameter, measuring the strength of the quadratic instability. Equation (1.2) is also derived, when a general nonlinearity is expanded via Taylor's formula. The dynamics of (1.2) was studied in [CH93], [HMBD95], [BK06] and recently [BD11] among others. In these articles the usual approach of amplitude equations is the derivation of a simplified model in the vicinity of the change of stability at $r = 0$. Both (1.1) and (1.2) are in the case of ϑ not being too large very well approximated by

$$u(t,x) \approx \sqrt{|r|}\cdot A(|r|t)\cdot e^{ix} + \sqrt{|r|}\cdot \overline{A(|r|t)}\cdot e^{-ix}, \tag{1.3}$$

where the complex amplitude $A(T)$ of the dominant frequency e^{ix} is the solution of

$$\partial_T A = \operatorname{sgn}(r)A + 3(\tfrac{38}{27}\vartheta^2 - 1)|A|^2 A, \tag{1.4}$$

which is accordingly named amplitude equation (AE, for short) of (1.2). Note that $T = |r|t$ denotes the slow time. As throughout this work the overline indicates the complex conjugate.

For the deterministic Swift-Hohenberg equation on an unbounded domain solutions are approximated via the Ginzburg-Landau PDE, as a whole band of uncountably many eigenvalues changes stability. For more results on the deterministic Swift-Hohenberg equation, see for instance [KSM$^+$92], [CE90], [MSZ00] and [Sch96].

It is the aim of this thesis to provide rigorous error estimates and to verify the existence of an amplitude equation for equation (1.2) with added noise which is

Parts of this thesis have been published in [KBM13] constant in space. That is, for the following stochastic generalised Swift-Hohenberg equation:

$$\partial_t u = \nu\varepsilon^2 u - (1+\Delta)^2 u + \vartheta u^2 - u^3 + \varepsilon\sigma\partial_t\beta, \tag{SH}$$

where $\beta(t)$ is a real valued standard Brownian motion and ϑ, σ and ν are real-valued constants. The small parameter $\varepsilon > 0$ relates the distance from bifurcation to the noise strength.

To be more precise we show that depending on the spatial domain (SH) is well approximated by

- For bounded domains:

$$u(t,x) \approx \varepsilon A(\varepsilon^2 t) \cdot e^{ix} + \varepsilon\overline{A(\varepsilon^2 t)} \cdot e^{-ix} \ ,$$

  where the complex-valued amplitude $A(T)$ solves the Itô differential equation

$$dA = (\nu A + 3(\tfrac{38}{27}\vartheta^2 - 1)A|A|^2 + 3(\vartheta^2 - \tfrac{1}{2})\sigma^2 A)dT + 2\vartheta\sigma A d\tilde{\beta}. \tag{1.5}$$

- For the unbounded domain:

$$u(t,x) \approx \varepsilon A(\varepsilon^2 t, \varepsilon x) \cdot e^{ix} + \varepsilon\overline{A(\varepsilon^2 t, \varepsilon x)} \cdot e^{-ix} \ ,$$

  where the amplitude $A(T,X)$ solves the Itô partial differential equation

$$dA = (4\partial_X^2 A + \nu A + 3(\tfrac{38}{27}\vartheta^2 - 1)A|A|^2 + 3(\vartheta^2 - \tfrac{1}{2})\sigma^2 A)dT + 2\vartheta\sigma A d\tilde{\beta}. \tag{1.6}$$

Here $\tilde{\beta}(T) := \varepsilon\beta(\varepsilon^{-2}T)$ is a rescaled version of $\beta(t)$.

Of course different scalings of noise strength are possible, but then in the amplitude equation, either the noise or the linear term disappears.

We show that in our scaling, although the constant mode is non-dominant, the noise appears also in the amplitude equation through coupling between Fourier modes (or parts of the frequency spectrum for the real line) induced by the nonlinear terms. Additional terms on the right-hand side are created and the noise appears multiplicative.

Within the same framework, we could treat several similar kinds of spatial noise not acting on the dominant part of the spectrum directly. On the other hand if we would add additive noise that acts on the dominant frequencies, then we would need to change scaling and consider smaller noise. See for example [BHP07] or [Blö07]. We give more details on this in Chapter 7. Let us emphasise that our approach does not cover thermal noise, but only x-independent perturbations acting uniformly on the whole system.

It is an interesting observation, that the amplitude equation contains only multiplicative noise instead of additive noise. This is due to the fact that the noise acting not directly on the dominant frequencies is mapped by the (in this case quadratic) nonlinearity back to the dominant frequencies. In order to obtain additive noise only, one would need to force directly the dominant frequencies. But in that case as described above it would be essential to have smaller noise in order to get a meaningful result. Other work in this direction can be found in [BH04] for Swift-Hohenberg without quadratic terms or [BM09] for an equation of Burgers type.

In the case of the bounded domain we focus, for simplicity of presentation, only on one specific example of boundary conditions and consider (SH) with periodic boundary conditions on $[0, 2\pi]$ only. Dirichlet and Neumann conditions would yield similar results. More details on other types of boundary conditions are also found in Chapter 7.

Another interesting observation is that due to our choice of the quadratic nonlinearity unstable terms both cubic and linear arise in the amplitude equation. The additional terms arise from nonlinear interaction, where squares of the noise actually average to a constant. This is significantly different to other quadratic nonlinearities like Burgers, for example, where these terms are all stabilising. See [BHP07] for a rigorous treatment of a large class of equations that contain the Burgers equation and [BMNW12] for numerical experiments, showing that in some cases the approximation remains true for surprisingly long times.

Our research on noise induced stabilisation was initiated originally by the observations of Axel Hutt and collaborators [HLSG07], who treated the case with $\vartheta = 0$. By numerical simulations of the equation on very large domains and formal arguments based on the non-rigorous application of centre manifold theory they derived the amplitude equation for the standard Swift-Hohenberg equation with noise constant in space. Moreover, they pointed out that additive noise has the potential to stabilise the dynamics. We revisit rigorous results in this direction in Chapter 3.

A similar stabilisation effect of the Burgers equation was also observed by A. Roberts [Rob03] with a single noise forcing the sine-Fourier mode. This was later established rigorously in [BHP07], even in the case of higher dimensional noise. The main difference for Burgers is that the amplitude equation still contains multiplicative noise, while in the situation of standard Swift-Hohenberg no noise remains in the amplitude equation. It only acts on higher order correction terms.

The case of quadratic nonlinearities (as in the Burgers equation) is much more involved than the case of cubic nonlinearities (as in the standard Swift-Hohenberg equation). For quadratic nonlinearities the interaction of the noise and the nonlinearity complicates the problem significantly, as non-dominant frequencies have a significant impact on the dominant part of the spectrum, see [BHP07]. While for cubic nonlinearities which were studied in [BM13] only the dominant modes survive and all other Fourier modes are treated as error terms. Additional terms only arise due to averaging of noise and nonlinear interaction of noise.

The thesis is organised as follows: Chapter 2 provides basic definitions. Chapter 3 recalls earlier results for the standard Swift-Hohenberg equation. In Chapter 4 & 5 we state and prove the main approximation results for bounded domains and the unbounded domain respectively. The result and proof for the bounded domain are more detailed versions of the ones published in [KBM13], where some errors have also been corrected. Additionally we give a heuristic overview over the structure of the linear operator, its interaction with the nonlinearity and the resulting solution. Chapter 6 contains short proofs for the assumed existence of solutions for the generalised Swift-Hohenberg equation and Chapter 7 discusses possible extensions of results. In Appendix A we collect some proofs for the boundedness with respect to the supremum norm of the semigroups occurring in this thesis.

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2 Setting

We consider mild solutions (defined below) of (SH) with values in the spaces

$$L^2_{per} = L^2_{per}([0, 2\pi]),$$

i.e. the space of 2π periodic $L^2[0, 2\pi]$ integrable functions which is isomorphic to $L^2[0, 2\pi]$, and for $\alpha > 1/2$

$$\mathcal{H}^\alpha \oplus \mathbb{R} = \{u + c : u \in \mathcal{H}^\alpha(\mathbb{R}), c \in \mathbb{R}\},$$

where $\mathcal{H}^\alpha$ is the family of fractional Sobolev spaces defined in terms of the Fourier transform

$$\mathcal{F}(u)(k) := \tfrac{1}{2\pi} \int_{\mathbb{R}} u(x) e^{-ikx} dx.$$

Definition 1. *Let $\alpha \geq 0$ and define the weight $\zeta_\alpha(k) := (1 + k^2)^{\alpha/2} \in C^\infty(\mathbb{R})$. Then the set*

$$\mathcal{H}^\alpha := \{u \in L^2(\mathbb{R}; \mathbb{C}) : \mathcal{F}^{-1}(\zeta_\alpha \mathcal{F} u) \in L^2(\mathbb{R}; \mathbb{C})\},$$

is a Hilbert space with the norm

$$\|u\|_{\mathcal{H}^\alpha} := \|\zeta_\alpha \mathcal{F} u\|_{L^2}.$$

Remark 2. *The space $\mathcal{H}^\alpha \oplus \mathbb{R}$ is a Hilbert space with scalar product*

$$\langle u_1 + c_1, u_2 + c_2 \rangle = c_1 c_2 + \langle u_1, u_2 \rangle_{\mathcal{H}^\alpha}$$

and the Fourier transform extends to $\mathcal{H}^\alpha \oplus \mathbb{R}$ in a natural way by defining

$$\mathcal{F}(1) := \delta_0,$$

where δ_0 denotes the Dirac distribution in zero.

Obviously for any $v \in \mathcal{H}^\alpha \oplus \mathbb{R}$ the splitting $v = u + c$ is unique and always exists as it can be obtained through the projection

$$P_c(u + c) := \operatorname*{ess.lim}_{x \to \infty} (u(x) + c) = c, \tag{2.1}$$

where the limit stands for the essential limit

$$\operatorname*{ess.lim}_{x \to \infty} u(x) := \lim_{n \to \infty} \int_n^{n+1} u(x) dx.$$

Before we define mild solutions on these spaces we introduce the semigroups

$$
\begin{aligned}
S_{[0,2\pi]}(t) : L^2_{per} \to L^2_{per},\, u(x) &\mapsto \sum_{k\in\mathbb{N}} e^{-t(1-k^2)^2} \hat{u}_k e^{ikx} \\
S_{\mathbb{R}}(t) : \mathcal{H}^\alpha \oplus \mathbb{R} \to \mathcal{H}^\alpha \oplus \mathbb{R},\, u(x) &\mapsto \int_{\mathbb{R}} e^{-t(1-k^2)^2} \mathcal{F}(u)(k) e^{ikx} dk,
\end{aligned}
\tag{2.2}
$$

where $\hat{u}_k$ is the k-th Fourier coefficient of $u \in L^2_{per}$ defined by

$$
\hat{u}_k := \tfrac{1}{2\pi} \int_0^{2\pi} u(x) e^{-ikx} dx. \tag{2.3}
$$

One can easily verify that these equations define a semigroup on the respective space and are generated by

$$
-\mathcal{L} := -(1+\partial_x^2)^2,
$$

in the first case with periodic boundary conditions and in the second case on the whole real line. Also we have $\lim_{t\downarrow 0} S_{[0,2\pi]}(t)u = u$ for all $u \in L^2_{per}$ respectively $\lim_{t\downarrow 0} S_{[0,2\pi]}(t)u = u$ for all $u \in \mathcal{H}^\alpha \oplus \mathbb{R}$ so by definition $S_{[0,2\pi]}$ and $S_{\mathbb{R}}$ are C_0 semigroups. For further details on semigroups see [Paz83].

Definition 3. *A stochastic process $u(t)$, $t \in [0,\tau]$ with continuous paths in $\mathcal{X} \in \{L^2_{per}, \mathcal{H}^\alpha \oplus \mathbb{R}\}$ is a mild solution of* (SH) *up to the stopping time τ if the following variation of constants formula holds in $\mathcal{X}$ for all $t \in [0,\tau]$:*

$$
\begin{aligned}
u(t) = e^{-t(1+\partial_x^2)^2} u(0) + \int_0^t e^{-(t-s)(1+\partial_x^2)^2} [\nu\varepsilon^2 u(s) + \vartheta u^2(s) - u^3(s)] ds \\
+ \varepsilon \int_0^t e^{-(t-s)(1+\partial_x^2)^2} \sigma d\beta(s)\ ,
\end{aligned}
\tag{2.4}
$$

where $e^{-t(1+\partial_x^2)^2}$ is the semigroup generated by $-\mathcal{L} : D(-\mathcal{L}) \subset \mathcal{X} \to \mathcal{X}$ i.e.

$$
e^{-t(1+\partial_x^2)^2} := \begin{cases} S_{[0,2\pi]} & \text{for } \mathcal{X} = L^2_{per}, \\ S_{\mathbb{R}} & \text{for } \mathcal{X} = \mathcal{H}^\alpha \oplus \mathbb{R}, \end{cases}
$$

with $S_{[0,2\pi]}, S_{\mathbb{R}}$ as defined in (2.2).

Independently of the chosen space $\mathcal{X} \in \{L^2_{per}, \mathcal{H}^\alpha \oplus \mathbb{R}\}$ the stochastic integral on the right-hand side of (2.4) can be simplified to

$$
Z(t) := \varepsilon\sigma \int_0^t e^{-(t-s)(1+\partial_x^2)^2} 1 d\beta(s) = \varepsilon\sigma \int_0^t e^{-(t-s)} d\beta(s), \tag{2.5}
$$

which is a simple real-valued Ornstein-Uhlenbeck process. If we rescale this process to the slow time we get the fast Ornstein-Uhlenbeck process defined by

$$
Z_\varepsilon(T) := \sigma \int_0^T e^{-\varepsilon^{-2}(T-s)} d\tilde{\beta}(s), \tag{2.6}
$$

where $\tilde{\beta}(T)$ is a rescaled version of the process defined by

$$\tilde{\beta}(T) := \varepsilon\beta(\varepsilon^{-2}T).$$

Using standard theory given in [DPZ92], it is straightforward to verify that mild solutions as defined above exist. This is, for example, done via Banach's fixed-point theorem for unique local solutions which are defined up to a stopping time. For completeness of presentation we provide a short sketch of the proof in Chapter 6. From energy estimates it can actually be shown that these are global solutions already, which are defined up to a deterministic time.
We do not rely on global existence in our results as we verify an approximation result up to stopping times where the mild solution gets too large and we use the amplitude equation to show that with very high probability this stopping time is greater than any fixed time.

Our approximation result measures the error in terms of the distance to the bifurcation point ($r = \nu = 0$) using big $\mathcal{O}$ notation modified for random variables. This is defined by the following:

Definition 4. *Let X_ε with $\varepsilon > 0$ be a family of stochastic variables in the normed vector space $\mathcal{X}$ and let $f(\varepsilon)$ be a function of ε. Then X_ε is of order $f(\varepsilon)$, which we abbreviate by*

$$X_\varepsilon = \mathcal{O}(f(\varepsilon)),$$

if and only if for every p-th moment of X_ε there is a constant C_p such that the following is valid for all $\varepsilon > 0$:

$$\mathbb{E}\left(\|X_\varepsilon\|_{\mathcal{X}}^p\right) \le C_p|f(\varepsilon)|^p.$$

The time $T_0 > 0$ does, throughout the whole work, stand for a time that is fixed a priori and, in particular, independent of ε.

Constants depending on T_0 and initial conditions but independent of the time $t \le T_0$ and ε will generally be denominated by the letter C or c. The value may change during calculation steps but this is used sparsely to hopefully avoid any confusions.

3 The case of the standard Swift-Hohenberg equation ($\vartheta = 0$)

The special case without quadratic nonlinearity has been studied in [BM13] for bounded domains and [MBK13] on the unbounded domain. As we will see these results, which we reproduce here, are consistent with the approximation theorems presented in Chapter 4 and Chapter 5.

First we apply Theorem 9 in [BM13] on the setting of (SH) as in Section 6.2.1 of the same article:

Proposition 5. *Let $0 < \kappa < \frac{1}{36}$. Let u be a stochastic process with continuous paths in L^2_{per} that is a mild solution of* (SH) *with $\vartheta = 0$ and $\|u(0)\|_{\mathcal{H}^1} = \mathcal{O}(\varepsilon^{1-\kappa})$, where the norm here means (with $\hat{u}_k$ as defined by equation* (2.3)*)*

$$\|u\|^2_{\mathcal{H}^1} = \sum_{k\in\mathbb{Z}} (1 + k^2)|\hat{u}_k|^2.$$

Then for every $p \in \mathbb{N}$ there is a constant C_p such that the rescaled solution

$$v(t, x) = \varepsilon^{-1} u(\varepsilon^{-2} t, x)$$

satisfies

$$\mathbb{P}\big(\|v(T, x) - \gamma_1(T)\sin(x) - \gamma_{-1}(T)\cos(x) - \varepsilon\sigma d\beta(T)\|_{\mathcal{H}^1} \geq \varepsilon^{1-38\kappa}\big) \leq C_p\varepsilon^p,$$

where γ_1 and γ_{-1} are the solutions of the following two-dimensional amplitude equations:

$$d\gamma_i = (\nu - 3\sigma^2)\gamma_i dT - \tfrac{3}{4}\gamma_i(\gamma_1^2 + \gamma_{-1}^2)dT \qquad \text{for } i = \pm 1.$$

By setting $\gamma := (\gamma_{-1} - i\gamma_1)/2$ we get

$$d\gamma = ((\nu - 3\sigma^2)\gamma - 3\gamma|\gamma|^2)dT,$$

which is the equivalent to Equation (1.5).

Our approximation result presented in Chapter 4 uses the supremum norm instead of the $\mathcal{H}^1$ norm provided here, which makes the proof slightly more involved. It would be no problem however to establish our result with the $\mathcal{H}^1$ norm by using similar methods as we use for our proof in Section 4.3.

For the unbounded domain we directly cite Theorem 3.4 from [MBK13]:

Proposition 6. *Let $u(t,x) \in C([0,T_0], L^\infty(\mathbb{R}))$ be a mild solution of* (SH) *with $\vartheta = 0$ and $A(T,X)$ be a solution of* (1.6) *such that $A \in C([0,T_0], \mathcal{H}^\alpha(\mathbb{R}))$ for $\alpha > \frac{1}{2}$.*

Assume, with the formal approximation $w_A(T,X)$ defined by

$$w_A(T,X) = A(T,X)e^{i\varepsilon^{-1}X} + \overline{A(T,X)}e^{-i\varepsilon^{-1}X},$$

that the initial conditions fulfil

$$\|u(0,x) - \varepsilon w_A(0,\varepsilon x)\|_\infty \le d\varepsilon^{1-3\kappa_0}\phi_\varepsilon$$

for some fixed $d > 0$ and for $\kappa_0 \in (0, \frac{1}{8})$ such that $\varepsilon^{-8\kappa_0}\phi_\varepsilon^2 \to 0$ for $\varepsilon \to 0$, where

$$\phi_\varepsilon^2 = \begin{cases} \varepsilon^2 & \text{if } \alpha > 3/2, \\ \varepsilon^2 \ln(1/\varepsilon) & \text{if } \alpha = 3/2, \\ \varepsilon^{2\alpha-1} & \text{if } \alpha < 3/2. \end{cases}$$

Then for every $p \in \mathbb{N}$ there exists a constant $C_p > 0$ depending on $\sup_{[0,T_0]}\|A\|_\alpha$, such that

$$\mathbb{P}\Big\{ \sup_{t\in[0,\varepsilon^{-2}T_0]} \big\|u(t,x) - \varepsilon w_A(\varepsilon^2 t, \varepsilon x) - \varepsilon Z_\varepsilon(\varepsilon^2 t)\big\|_\infty > C\varepsilon^{1-4\kappa_0}\phi_\varepsilon \Big\} \le C_p\varepsilon^p, \tag{3.1}$$

where $Z_\varepsilon(T)$ is the fast Ornstein-Uhlenbeck process defined in (2.6).

This result also fits nicely with the proposed amplitude equation (1.6). It does not put any assumptions on the spectral shape of the initial condition which, in return, leads to a more coarse approximation as the one we will show in Chapter 5.

4 Result for the bounded domain

4.1 Approximation theorem for the bounded domain

The first main result is the following approximation theorem for the stochastic generalised Swift-Hohenberg equation (SH), which was published in [KBM13].

Theorem 7. *Let $T_0 > 0$ be a time of order 1, $\vartheta \in \mathbb{R}$ with $\vartheta^2 \leq \frac{27}{38}$ and $0 < \kappa < \frac{1}{27}$. Let u be a stochastic process with continuous paths in $L^2_{per}[0, 2\pi]$ that is a mild solution of* (SH) *with $\|u(0)\|_\infty = \mathcal{O}(\varepsilon^{1-\kappa})$. Furthermore, let $A(T)$, $T \in [0, T_0]$ be a stochastic process with continuous paths in $\mathbb{C}$ that solves* (1.5) *with*

$$A(0) = \frac{1}{2\pi} \int_0^{2\pi} \varepsilon^{-1} u(0, x) e^{ix} dx \; = \mathcal{O}(\varepsilon^{-\kappa}). \tag{4.1}$$

Then for all $p \in \mathbb{N}$ there is a constant C_p such that the following holds:

$$\mathbb{P}\Big(\sup_{t \in [0, \varepsilon^{-2} T_0]} \|u(t) - u_A(t) - \varepsilon Z_\varepsilon(\varepsilon^2 t) - e^{-t(1+\partial_x^2)^2} u_s(0)\|_\infty > \varepsilon^{2-29\kappa} \Big) \leq C_p \varepsilon^p, \tag{4.2}$$

with the approximation

$$u_A(t, x) = \varepsilon A(\varepsilon^2 t) e^{ix} + \varepsilon \overline{A(\varepsilon^2 t)} e^{-ix},$$

where Z_ε is the Ornstein-Uhlenbeck process defined by

$$Z_\varepsilon(T) := \varepsilon^{-1} \sigma \int_0^T e^{-\varepsilon^{-2}(T-s)} d\tilde{\beta}(s). \tag{4.3}$$

Here we easily see that $Z_\varepsilon(\varepsilon^2 t) = Z(t)$ with Z defined in (2.5).

The small constant $\kappa > 0$ introduced in the theorem above mainly takes care of the fact, that we cannot bound the fast stochastic convolution Z_ε uniformly in time by a constant with high probability, but by a bound that is slightly worse depending on ε. As the final error bound will thus be slightly worse than order $\mathcal{O}(\varepsilon^2)$, we can also allow for initial conditions $u(0)$ that are not of order $\mathcal{O}(\varepsilon)$, but slightly worse.

Although we think of κ being very small, the previous theorem is still true for $\kappa \in [1/29, 1/27]$, but useless, as we loose a full order of ε in the final result.

Remark 8. *In the case $\vartheta^2 > \frac{27}{38}$, which is not treated in Theorem 7, the amplitude equation* (1.5) *has an unstable cubic nonlinearity, and thus exhibits blow up in finite time, while in the case $\vartheta^2 = \frac{27}{38}$* (1.5) *loses the cubic completely.*

Nevertheless as long as the solution A to (1.5) *is not too large (for example $|A(T)| \leq \varepsilon^{-2\kappa}$) our approximation result still holds, up to a stopping time, where A fails to be bounded.*

The proof is basically the same except the fixed time $T_0 > 0$ is replaced everywhere by the stopping time $\tau_A = \inf\{t : |A(t)| \geq \varepsilon^{-2\kappa}\} \wedge T_0$. For simplicity of presentation, we refrain from giving more details here.

Remark 9. *The interesting case $\vartheta^2 = \frac{27}{38}$ was studied in the deterministic case. See for example [BD11], where an even more general case was treated. In this case* (1.5) *loses its cubic nonlinearity, and turns out to be a linear equation only. Thus we can consider larger solutions and hence larger noise. By changing the scaling still a meaningful amplitude equation is obtained but now with a quintic nonlinearity.*

Using the methods presented in this chapter it is straightforward but lengthy to derive the quintic amplitude equation also in the stochastic case. We refrain from giving details here.

4.2 Heuristics

If we look at the spectrum of the operator $-\mathcal{L} + \varepsilon^2\nu$, shown in Figure 4.1 as a schematic representation, we see that only the first Fourier modes have a small positive eigenvalue of order ε^2 while all others are negative with an absolute value bounded away from zero. This means that in the solution to the linear equation

$$u(t) = e^{-t(\mathcal{L}+\varepsilon^2\nu)}u(0)$$

all modes except the first ones are pushed down with speed one and the first modes become dominant with slowly moving Fourier coefficients. With added nonlinearity we see the same effect when starting with an initial value small enough such that the nonlinearity does not dominate the semigroup.

To get a better understanding we can change our point of view by rescaling $u(t, x)$ to the appropriate slow time-scale by setting

$$v(T, x) := \varepsilon^{-1}u(\varepsilon^{-2}T, x) .$$

Then equation (SH) becomes

$$dv = (-\varepsilon^{-2}(1 + \partial_x^2)^2 v + \nu v + \varepsilon^{-1}\vartheta v^2 - v^3)dT + \varepsilon^{-1}\sigma d\tilde{\beta} .$$

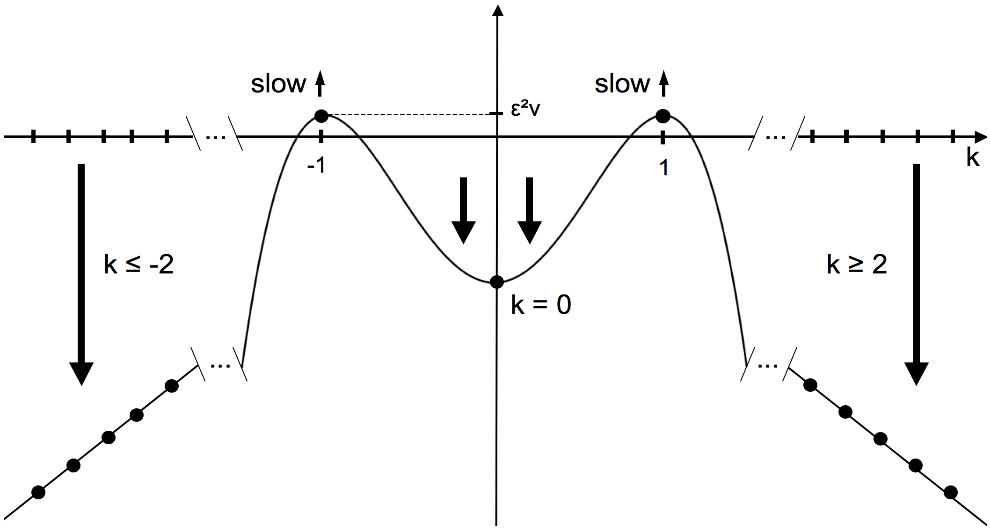

Figure 4.1 Spectrum of the operator $-\mathcal{L}+\varepsilon^2\nu$ on $L^2_{per}[0,2\pi]$

With an initial value $v(0)$ of order 1 the nonlinear terms are of order ε^{-1} and as we will see the non-dominant modes get pushed down to being of order ε except for the stochastic term on the constant mode. This leads to the situation shown in Figure 4.2. The figure contains all interactions of modes through the nonlinearity that are relevant to the shape of the amplitude equation (1.5): Terms of order ε^{-1} or larger acting on the non-dominant modes $\{\dots,-2,0,+2,\dots\}$ and terms of order one or larger acting on the dominant modes $\{-1,+1\}$.

These nonlinear interactions of the noise together with averaging (see Lemma 13) lead to some surprising deterministic terms in the amplitude equation. A stabilising linear term from the cubic term, that was already observed in [Hut08], and destabilising terms both cubic and linear that arise from the quadratic term. But if ϑ is not too large, increasing the noise strength σ may lead to a stabilisation effect.

4.3 Proof of the result

4.3.1 Preliminaries

We start out by rescaling $u(t,x)$ to the slow time-scale by

$$v(T,x) := \varepsilon^{-1}u(\varepsilon^{-2}T,x)\ .$$

Its stochastic differential is given by

$$dv = (-\varepsilon^{-2}(1+\partial_x^2)^2v + \nu v + \varepsilon^{-1}\vartheta v^2 - v^3)dT + \varepsilon^{-1}\sigma d\tilde{\beta}\ .$$

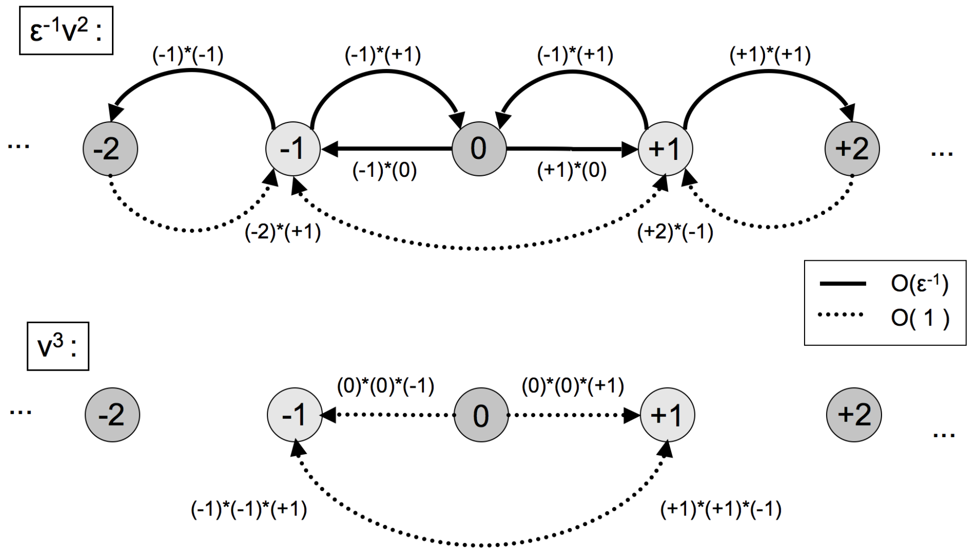

Figure 4.2 Relevant interactions between modes through the nonlinearity

The mild formulation is:

$$\begin{aligned} v(T) &= e^{-T\varepsilon^{-2}(1+\partial_x^2)^2} v(0) + Z_\varepsilon(T) \\ &\quad + \int_0^T e^{-(T-s)\varepsilon^{-2}(1+\partial_x^2)^2} [\nu v(s) + \varepsilon^{-1}\vartheta v^2(s) - v^3(s)] ds \; . \end{aligned} \tag{4.4}$$

Here Z_ε is the fast Ornstein-Uhlenbeck process defined in (4.3). It is the solution of

$$dZ_\varepsilon = -\varepsilon^{-2} Z_\varepsilon dT + \sigma \varepsilon^{-1} d\tilde{\beta}, \quad Z_\varepsilon(0) = 0 \; . \tag{4.5}$$

Also we define the stopping time

$$\tau^* = \inf \{T > 0 : \|v(T)\|_\infty > \varepsilon^{-\kappa_0}\} \wedge T_0, \tag{4.6}$$

where κ is defined in Theorem 7 and κ_0 is any small real value with $\kappa_0 > 2\kappa$, which asserts that $\tau^* > 0$ almost surely. Later we fix $\kappa_0 = \frac{17}{8}\kappa$ in the proof of Theorem 7. Expanding $v(T,x)$ as a complex Fourier series yields

$$v(T,x) = \sum_{k=-\infty}^{\infty} \hat{v}_k(T) e^{ikx} \; . \tag{4.7}$$

Define a splitting of the Fourier modes into the non-dominant modes

$$v_s(T,x) = \sum_{|k|\neq 1} \hat{v}_k(T) e^{ikx} \tag{4.8}$$

and the dominant modes

$$v_c(T,x) = v(T,x) - v_s(T,x) = \hat{v}_1(T) e^{ix} + \overline{\hat{v}_1(T)} e^{-ix} \; . \tag{4.9}$$

For technical reasons we also define

$$v_\infty(T,x) = \sum_{|k|\geq 3} [\hat{v}_k(T) - e^{-T\varepsilon^{-2}(1-k^2)^2}\hat{v}_k(0)]e^{ikx} \tag{4.10}$$

For $|k| \geq 1$ from the mild solution (4.4), each $\hat{v}_k$ is given by

$$\begin{aligned} \hat{v}_k(T) &= e^{-\varepsilon^{-2}(1-k^2)^2T}\hat{v}_k(0) \\ &\quad + \int_0^T e^{-\varepsilon^{-2}(1-k^2)^2(T-s)} \left[\nu\hat{v}_k(s) + \varepsilon^{-1}\vartheta\widehat{(v^2)}_k(s) - \widehat{(v^3)}_k(s)\right] ds, \end{aligned} \tag{4.11}$$

where the hat indicates the discrete Fourier transform and the lower index k denotes its k-th mode.

Later we will need that the semigroup $e^{-t\mathcal{L}}$ acting on the space $L^2_{per}[0,2\pi]$ is bounded on $L^\infty_{per}[0,2\pi]$. We prove the following in Appendix A:

Corollary A.3. *Let $e^{-t\mathcal{L}}$ be the semigroup on $L^2_{per}[0,2\pi]$ defined by*

$$e^{-t\mathcal{L}}u(x) = \sum_{k\in\mathbb{N}} e^{-t(1-k^2)^2}\hat{u}_k e^{ikx}.$$

Then there exists a constant $C > 0$ such that for all $t \geq 0$:

$$\|e^{-t\mathcal{L}}u\|_\infty \leq C\|u\|_\infty.$$

4.3.2 Removing non-dominant modes

Next we show that the non-dominant modes ($|k| \neq 1$) of the rescaled solution $v(t)$ defined in the last section can be approximated by the fast OU-process Z_ε. With a slight abuse of the $\mathcal{O}$-notation, our result states:

$$v_s(T) = e^{-T\varepsilon^{-2}(1+\partial_x^2)^2}v_s(0) + Z_\varepsilon(T) + \mathcal{O}(\varepsilon^{1-2\kappa_0}) .$$

Or, to be more precise:

Lemma 10. *Under the assumptions of Theorem 7, with stopping time τ^* defined by (4.6) and $\hat{v}_k$ as in (4.7), the following statements are true:*

$$\sup_{T\in[0,\tau^*]} \|\sum_{|k|\geq 2} [\hat{v}_k(T) - e^{-T\varepsilon^{-2}(1-k^2)^2}\hat{v}_k(0)] \cdot e^{ikx}\|_\infty = \mathcal{O}(\varepsilon^{1-2\kappa_0}) , \tag{4.12}$$

$$\sup_{T\in[0,\tau^*]} \|\hat{v}_0(T) - Z_\varepsilon(T) - e^{-T\varepsilon^{-2}}v_0(0)\| = \mathcal{O}(\varepsilon^{1-2\kappa_0}) . \tag{4.13}$$

Proof. Since $\|v\|_\infty \leq \varepsilon^{-\kappa_0}$, it follows that for any $k \in \mathbb{Z}$ and $n \in \mathbb{N}$

$$|(\widehat{v^n})_k| \leq \Big(\sum_{k\in\mathbb{Z}} |(\widehat{v^n})_k|^2\Big)^{1/2} = \|\widehat{v^n}\|_{\ell^2} = \|v^n\|_{L^2_{per}} \leq \sqrt{2\pi}\|v^n\|_\infty \leq \sqrt{2\pi}\varepsilon^{-n\kappa_0}. \quad (4.14)$$

In combination with the simple inequality (for $|k| \neq 1$)

$$\int_0^T e^{-\varepsilon^{-2}(1-k^2)^2(T-s)} ds \leq (1-k^2)^{-2}\varepsilon^2,$$

we derive the following by bounding the integral term in (4.11)

$$\Big|\hat{v}_k(T) - e^{-T\varepsilon^{-2}(1-k^2)^2}\hat{v}_k(0)\Big| \leq \varepsilon^{1-2\kappa_0}(1-k^2)^{-2}(2+|\nu|+|\vartheta|). \quad (4.15)$$

Therefore with $\sum_{|k|\geq 2}(1-k^2)^{-2} \leq \sum_{k=1}^\infty k^{-2} = \frac{\pi^2}{6}$ we obtain (using $\kappa_0 < 1$ for the cubic term)

$$\sum_{|k|\geq 2}\Big|\hat{v}_k(T) - e^{-T\varepsilon^{-2}(1-k^2)^2}\hat{v}_k(0)\Big| \leq \varepsilon^{1-2\kappa_0}\frac{\pi^2}{3}(2+|\nu|+|\vartheta|),$$

which proves (4.12). Projecting the mild solution (2.4), the constant mode $\hat{v}_0$ has the form

$$\begin{aligned}\hat{v}_0(T) =& e^{-\varepsilon^{-2}T}\hat{v}_0(0) + Z_\varepsilon(T) \\ &+ \int_0^T e^{-\varepsilon^{-2}(T-s)}(\nu v_0(s) + \varepsilon^{-1}\vartheta(\widehat{v^2})_0(s) - (\widehat{v^3})_0(s))ds\,.\end{aligned} \quad (4.16)$$

Thus with similar arguments as before, for all $T < \tau^*$ the left-hand side of (4.13) is bounded by

$$\Big|\hat{v}_0(T) - Z_\varepsilon(T) - e^{-\varepsilon^{-2}T}\hat{v}_0(0)\Big| \leq \varepsilon^{1-2\kappa_0}(2+|\nu|+|\vartheta|).$$

□

4.3.3 Rewriting the first Fourier-Mode

We continue by showing that the dominant mode $\hat{v}_1(T)$ is well approximated by $A(T)$. For simplicity of presentation let us define the following functions:

$$a(T) := \hat{v}_1(T), \qquad \Phi(T) := \varepsilon^{-1}\Big(\hat{v}_2(T) - e^{-9T\varepsilon^{-2}}\hat{v}_2(0)\Big),$$
$$\Psi(T) := \varepsilon^{-1}\Big(\hat{v}_0(T) - Z_\varepsilon(T) - e^{-T\varepsilon^{-2}}\hat{v}_0(0)\Big).$$

Lemma 11. *Under the assumptions of Lemma 10, the stochastic differential of $a(T)$ is given by*

$$da = (\nu a + 3(\tfrac{38}{27}\vartheta^2 - 1)a|a|^2 + 6(\vartheta^2 - \tfrac{1}{2})aZ_\varepsilon^2)dT + 2\vartheta\sigma a d\tilde{\beta} + dR, \tag{4.17}$$

where $R(t)$ is a stochastic process with $\sup_{t\in[0,\tau^*]}|R(t)| = \mathcal{O}(\varepsilon^{1-8\kappa_0})$.

Proof. In Lemma 10 in (4.14) and (4.15) we established:

$$\sup_{T\in[0,\tau^*]} |\hat{v}_1(T)| \leq \varepsilon^{-\kappa_0} \tag{4.18}$$

$$\sup_{T\in[0,\tau^*]} \big(\sup_{|k|\geq 2} |\hat{v}_k(T) - e^{-\varepsilon^{-2}(1-k^2)^2}\hat{v}_k(0)|\big) = \mathcal{O}(\varepsilon^{1-2\kappa_0}). \tag{4.19}$$

This readily implies

$$\sup_{T\in[0,\tau^*]} |a(T)| = \mathcal{O}(\varepsilon^{-\kappa_0}), \quad \sup_{T\in[0,\tau^*]} |\Phi(T)| = \mathcal{O}(\varepsilon^{-2\kappa_0}), \quad \sup_{T\in[0,\tau^*]} |\Psi(T)| = \mathcal{O}(\varepsilon^{-2\kappa_0}).$$

The slightly better bound on a unfortunately does not improve the final result. We could just bound all three terms by $\mathcal{O}(\varepsilon^{-2\kappa_0})$.

The infinite-dimensional part is bounded by

$$\sup_{T\in[0,\tau^*]} \|v_\infty(T)\|_\infty = \mathcal{O}(\varepsilon^{1-2\kappa_0}). \tag{4.20}$$

The OU-process can be bounded by

$$\sup_{T\in[0,\tau^*]} |Z_\varepsilon(T)| = \mathcal{O}(\varepsilon^{-\zeta}) \tag{4.21}$$

for all positive $\zeta > 0$. For a proof of this well-known result see for example [BM13] p. 9 (Lemma 14).

Now we can directly calculate the stochastic differentials $da, d\Phi$ and $d\Psi$ by writing v as

$$v = ae^{ix} + \varepsilon\Phi e^{i2x} + \bar{a}e^{-ix} + \varepsilon\bar{\Phi}e^{-i2x} + \varepsilon\Psi + Z_\varepsilon + v_\infty + e^{-T\varepsilon^{-2}(1+\varepsilon^2\partial_x^2)^2}v_s(0)$$

and multiplying it with itself to bound $(\widehat{v^2})_k$ and $(\widehat{v^3})_k$ for $k \in \{0,1,2\}$. Note that we can bound the Fourier transform by the L^∞ norm. We have

$$\begin{aligned} v^2 &= 2(ae^{ix} + \bar{a}e^{-ix} + Z_\varepsilon)(\varepsilon\Phi e^{i2x} + \varepsilon\bar{\Phi}e^{-i2x} + \varepsilon\Psi + v_\infty) \\ &\quad + (ae^{ix} + \bar{a}e^{-ix} + Z_\varepsilon)^2 + r_1 \\ v^3 &= (ae^{ix} + \bar{a}e^{-ix} + Z_\varepsilon)^3 + r_2 \end{aligned} \tag{4.22}$$

with

$$\begin{aligned}
r_1 &= (\varepsilon\Phi e^{i2x} + \varepsilon\bar{\Phi}e^{-i2x} + \varepsilon\Psi + v_\infty)^2 + (e^{-\varepsilon^{-2}T\mathcal{L}}v_s(0))^2 \\
&\quad + 2(ae^{ix} + \varepsilon\Phi e^{i2x} + \bar{a}e^{-ix} + \varepsilon\bar{\Phi}e^{-i2x} + \varepsilon\Psi + Z_\varepsilon + v_\infty)e^{-\varepsilon^{-2}T\mathcal{L}}v_s(0) \\
r_2 &= (\varepsilon\Phi e^{i2x} + \varepsilon\bar{\Phi}e^{-i2x} + \varepsilon\Psi + v_\infty)^3 + (e^{-\varepsilon^{-2}T\mathcal{L}}v_s(0))^3 \\
&\quad + 3(ae^{ix} + \bar{a}e^{-ix} + Z_\varepsilon)(\varepsilon\Phi e^{i2x} + \varepsilon\bar{\Phi}e^{-i2x} + \varepsilon\Psi + v_\infty)^2 \\
&\quad + 3(ae^{ix} + \bar{a}e^{-ix} + Z_\varepsilon)^2(\varepsilon\Phi e^{i2x} + \varepsilon\bar{\Phi}e^{-i2x} + \varepsilon\Psi + v_\infty) \\
&\quad + 3(ae^{ix} + \bar{a}e^{-ix} + Z_\varepsilon)(e^{-\varepsilon^{-2}T\mathcal{L}}v_s(0))^2 + 3(ae^{ix} + \bar{a}e^{-ix} + Z_\varepsilon)^2(e^{-\varepsilon^{-2}T\mathcal{L}}v_s(0)) \\
&\quad + 3(\varepsilon\Phi e^{i2x} + \varepsilon\bar{\Phi}e^{-i2x} + \varepsilon\Psi + v_\infty)(e^{-\varepsilon^{-2}T\mathcal{L}}v_s(0))^2 \\
&\quad + 3(\varepsilon\Phi e^{i2x} + \varepsilon\bar{\Phi}e^{-i2x} + \varepsilon\Psi + v_\infty)^2(e^{-\varepsilon^{-2}T\mathcal{L}}v_s(0)) \\
&\quad + 6(ae^{ix} + \bar{a}e^{-ix} + Z_\varepsilon)(\varepsilon\Phi e^{i2x} + \varepsilon\bar{\Phi}e^{-i2x} + \varepsilon\Psi + v_\infty)(e^{-\varepsilon^{-2}T\mathcal{L}}v_s(0)).
\end{aligned}$$

Here we used for shorthand notation

$$\mathcal{L} = (1 + \partial_x^2)^2 \ .$$

Because of

$$\begin{aligned}
\sup_{T\in[0,\tau^*]} \|\varepsilon\Phi(T)e^{i2x} + \varepsilon\bar{\Phi}(T)e^{-i2x} + \varepsilon\Psi(T) + v_\infty(T)\|_\infty &= \mathcal{O}(\varepsilon^{1-2\kappa_0}), \\
\sup_{T\in[0,\tau^*]} \|a(T)e^{ix} + \bar{a}(T)e^{-ix} + Z_\varepsilon(T)\|_\infty &= \mathcal{O}(\varepsilon^{-\kappa_0}),
\end{aligned}$$

which follows from (4.18), (4.19), (4.20) and (4.21), together with

$$\begin{aligned}
\|\int_0^T e^{-\varepsilon^{-2}s\mathcal{L}}v_s(0)ds\|_\infty &\le \varepsilon^2 \sum_{|k|\neq 1}(1-k^2)^{-2}|(\widehat{v_s(0)})_k| \\
&\le \varepsilon^2\sqrt{2\pi}\sum_{|k|\neq 1}(1-k^2)^{-2}\|v_s(0)\|_\infty = \mathcal{O}(\varepsilon^{2-\kappa_0})
\end{aligned}$$

we can bound the integral in time of r_1 and r_2 by

$$\begin{aligned}
\sup_{T\in[0,\tau^*]}\|\int_0^T r_1 dt\|_\infty &= \mathcal{O}(\varepsilon^{2-6\kappa_0}) \\
\sup_{T\in[0,\tau^*]}\|\int_0^T r_2 dt\|_\infty &= \mathcal{O}(\varepsilon^{1-6\kappa_0}).
\end{aligned}$$

Analogously we can bound integrals of any power of $\|r_i\|_\infty$. Inserting (4.22) into the mild solution formulas (4.11) respectively (4.16) gives

$$da = (\nu a + 2\vartheta\bar{a}\Phi + 2\vartheta a\Psi - 3a|a|^2 - 3aZ_\varepsilon^2 + \varepsilon^{-1}2\vartheta aZ_\varepsilon + R_1)dT \tag{4.23}$$
$$d\Phi = (-9\varepsilon^{-2}\Phi + \varepsilon^{-2}\vartheta a^2 + R_2)dT \tag{4.24}$$
$$d\Psi = (-\varepsilon^{-2}\Psi + \varepsilon^{-2}\vartheta|a|^2 + \varepsilon^{-2}\vartheta Z_\varepsilon^2 + R_3)dT \tag{4.25}$$

where

$$R_1(t) = \varepsilon^{-1}\vartheta(\widehat{r_1})_1 - (\widehat{r_2})_1,$$
$$R_2(t) = \nu\Phi + 2\varepsilon^{-1}\vartheta Z_\varepsilon\Phi - 3\varepsilon^{-1}a^2 Z_\varepsilon + 2\varepsilon^{-2}\vartheta v_3\bar{a} + \varepsilon^{-2}\vartheta(\widehat{r_1})_2 - \varepsilon^{-1}(\widehat{r_2})_2$$

and

$$R_3(t) = \nu\Psi + \varepsilon^{-1}\vartheta\Psi Z_\varepsilon - \varepsilon^{-1}Z_\varepsilon^3 + 6\varepsilon^{-1}|a|^2 Z_\varepsilon + \varepsilon^{-2}\vartheta(\widehat{r_1})_0 - \varepsilon^{-1}(\widehat{r_2})_0$$

are stochastic processes with

$$\sup_{T\in[0,\tau^*]}\int_0^T |R_1|ds = \mathcal{O}(\varepsilon^{1-6\kappa_0}), \qquad \sup_{T\in[0,\tau^*]}\int_0^T |R_2| + |R_3|ds = \mathcal{O}(\varepsilon^{-1-6\kappa_0}).$$

In order to eliminate Φ and Ψ on the right-hand side of (4.23) we apply the Itô formula to $\bar{a}\Phi$, $a\Psi$ and aZ_ε. Note that there is no Itô correction at this point, as a, Φ, and Ψ are random but differentiable, which follows from the representation in (4.23) – (4.25).

$$\begin{aligned} d(\bar{a}\Phi) &= (d\bar{a})\Phi + \bar{a}(d\Phi) = (\bar{a}(-9\varepsilon^{-2}\Phi + \varepsilon^{-2}\vartheta a^2) + R_4)dT \\ d(a\Psi) &= (da)\Psi + a(d\Psi) = (a(-\varepsilon^{-2}\Psi + 2\varepsilon^{-2}\vartheta|a|^2 + \varepsilon^{-2}\vartheta Z_\varepsilon^2) + R_5)dT \\ d(aZ_\varepsilon) &= (da)Z_\varepsilon + a(dZ_\varepsilon) = (\varepsilon^{-1}2\vartheta a Z_\varepsilon^2 - \varepsilon^{-2}aZ_\varepsilon + R_6)dT + a\varepsilon^{-1}\sigma d\tilde{\beta} \end{aligned}$$

where

$$R_4(t) = \bar{a}R_2 + \Phi(\nu\bar{a} + 2\vartheta a\bar{\Phi} + 2\vartheta\bar{a}\bar{\Psi} - 3\bar{a}|a|^2 - 3\bar{a}Z_\varepsilon^2 + \varepsilon^{-1}2\vartheta\bar{a}Z_\varepsilon + \bar{R}_1),$$
$$R_5(t) = aR_3 + \Psi(\nu a + 2\vartheta\bar{a}\Phi + 2\vartheta a\Psi - 3a|a|^2 - 3aZ_\varepsilon^2 + \varepsilon^{-1}2\vartheta a Z_\varepsilon + R_1)$$

and

$$R_6(t) = Z_\varepsilon(\nu a + 2\vartheta\bar{a}\Phi + 2\vartheta a\Psi - 3a|a|^2 - 3aZ_\varepsilon^2 + R_1)$$

are stochastic processes with

$$\sup_{t\in[0,\tau^*]}\int_0^T |R_4| + |R_5|ds = \mathcal{O}(\varepsilon^{-1-8\kappa_0}), \qquad \sup_{t\in[0,\tau^*]}\int_0^T |R_6|ds = \mathcal{O}(\varepsilon^{-8\kappa_0}).$$

Therefore we have:

$$\bar{a}\Phi dT = (\frac{1}{9}\vartheta a|a|^2 + \frac{1}{9}\varepsilon^2 R_4)dT - \frac{1}{9}d(\varepsilon^2\bar{a}\Phi) \tag{4.26}$$
$$a\Psi dT = (2\vartheta a|a|^2 + \vartheta a Z_\varepsilon^2 + \varepsilon^2 R_5)dT - d(\varepsilon^2 a\Psi) \tag{4.27}$$
$$\varepsilon^{-1}aZ_\varepsilon dT = (2\vartheta a Z_\varepsilon^2 + \varepsilon R_6)dT + \sigma a d\tilde{\beta}(T) - d(\varepsilon a Z_\varepsilon). \tag{4.28}$$

By substituting (4.26) – (4.28) into (4.23) we get the desired result for da with

$$dR = 2\vartheta\varepsilon^2(\frac{1}{9}R_4 dT + R_5 dT - \frac{1}{9}d(\bar{a}\Phi) - d(a\Psi)) + 2\vartheta\varepsilon(R_6 dT - d(aZ)).$$

□

4.3.4 Averaging with error bounds

We match the equation for da from (4.17) to the amplitude equation (1.5). For this we need to remove $aZ_\varepsilon^2 dT$, which is done in this section. First we need the following technical Lemma:

Lemma 12. *Let X, f, g be $\mathbb{C}$-valued stochastic processes with*

$$X(t) = \int_0^t f(s)ds + \int_0^t g(s)d\tilde{\beta},$$

where $\sup_{t\in[0,\tau]}|f(t)| = \mathcal{O}(\varepsilon^\gamma)$ and $\sup_{t\in[0,\tau]}|g(t)| = \mathcal{O}(\varepsilon^\gamma)$ with $\gamma \in \mathbb{R}$ and $\tau \leq T_0$ being a stopping time. Then $X(t)$ has the same bound as $f(t)$ and $g(t)$:

$$\sup_{t\in[0,\tau]} |X(t)| = \mathcal{O}(\varepsilon^\gamma) \tag{4.29}$$

Proof. First we need to show that for any process $Y(t)$ we have the bounds

$$\mathbb{E}\Big(\sup_{t\in[0,\tau]} |\int_0^t Y d\tilde{\beta}|\Big)^p \leq C\Big(\mathbb{E}(\sup_{t\in[0,\tau]} |Y(t)|^{2p})\Big)^{1/2} \tag{4.30}$$

$$\mathbb{E}\Big(\sup_{t\in[0,\tau]} |\int_0^t Y ds|\Big)^p \leq C\Big(\mathbb{E}(\sup_{t\in[0,\tau]} |Y(t)|^{2p})\Big)^{1/2}, \tag{4.31}$$

where C is a positive constant depending on T_0 and p. With the Burkholder-Davis-Gundy inequality we obtain

$$\begin{aligned}\mathbb{E}\Big(\sup_{t\in[0,\tau]} |\int_0^t Y d\tilde{\beta}|\Big)^p &\leq C_p \mathbb{E}\Big(\Big[\int_0^\tau Y d\tilde{\beta}\Big]\Big)^{p/2} \\ &= C_p \mathbb{E}\Big(\int_0^\tau Y^2 [d\tilde{\beta}]\Big)^{p/2} = C_p \mathbb{E}\Big(\int_0^\tau Y^2 ds\Big)^{p/2},\end{aligned}$$

where C_p is a positive constant depending on p and $[\xi]$ denotes the quadratic variation of ξ. Similarly by using the Hölder inequality we get

$$\mathbb{E}\Big(\sup_{t\in[0,\tau]} |\int_0^t Y ds|\Big)^p \leq \mathbb{E}\Big(\sup_{t\in[0,\tau]} (t\int_0^t Y^2 ds)^{p/2}\Big) \leq (T_0)^{p/2}\mathbb{E}\Big(\int_0^\tau Y^2 ds\Big)^{p/2}.$$

Another two uses of the Hölder inequality prove (4.30) and (4.31):

$$\begin{aligned}\mathbb{E}\Big(\int_0^\tau Y^2 ds\Big)^{p/2} &\leq \mathbb{E}\Big((T_0)^{(p-1)}\int_0^\tau Y^{2p} ds\Big)^{1/2} \\ &\leq (T_0)^{(p-1)/2}\Big(\mathbb{E}\int_0^\tau (Y^{2p})ds\Big)^{1/2} \\ &\leq (T_0)^{(p-1)/2}\Big(T_0\mathbb{E}(\sup_{t\in[0,\tau]} |Y(t)|^{2p})\Big)^{1/2} \\ &\leq (T_0)^{p/2}\Big(\mathbb{E}(\sup_{t\in[0,\tau]} |Y(t)|^{2p})\Big)^{1/2}.\end{aligned}$$

With this we can bound $\sup_{t\in[0,\tau]}|X(t)|$:

$$\begin{aligned}
\mathbb{E}\sup_{t\in[0,\tau]}|X(t)|^p &\le \mathbb{E}(\sup_{t\in[0,\tau]}|\int_0^t f ds| + \sup_{t\in[0,\tau]}|\int_0^t g d\tilde{\beta}|)^p \\
&\le p\mathbb{E}(\sup_{t\in[0,\tau]}|\int_0^t f ds|^p + \sup_{t\in[0,\tau]}|\int_0^t g d\tilde{\beta}|^p) \\
&\le p\Big(\mathbb{E}(\sup_{t\in[0,\tau]}|\int_0^t f ds|)^{2p}\Big)^{1/2} + p\Big(\mathbb{E}(\sup_{t\in[0,\tau]}|\int_0^t g d\tilde{\beta}|)^{2p}\Big)^{1/2} \\
&\le pC\Big(\mathbb{E}(\sup_{t\in[0,\tau]}|f(t)|^{4p})\Big)^{1/4} + pC\Big(\mathbb{E}(\sup_{t\in[0,\tau]}|g(t)|^{4p})\Big)^{1/4}
\end{aligned}$$

where we used Young's inequality in the second step and the inequalities (4.30) and (4.31) in the last step. Since by the assumptions both $\sup_{t\in[0,\tau]}|f(t)|$ and $\sup_{t\in[0,\tau]}|g(t)|$ are of order ε^{γ}, our proof is finished. □

Now we can substitute the aZ^2 term in (4.17). This is done by using the averaging property of Z_ε described in the next Lemma.

Lemma 13. *Let $X(t) \in \mathbb{C}$ be a stochastic process with $dX = f(T)dT + g(T)d\tilde{\beta}$, where $\sup_{T\in[0,\tau]}|f(T)| = \mathcal{O}(\varepsilon^{-\gamma})$ and $\sup_{T\in[0,\tau]}|g(T)| = \mathcal{O}(\varepsilon^{-\gamma})$ with $\gamma > 0$ and $\tau \le T_0$ being a stopping time. Then with Z_ε as defined by* (4.5) *the following holds:*

$$\sup_{T\in[0,\tau]}|\int_0^T X(s)Z_\varepsilon(s)^2 ds - \int_0^T \tfrac{1}{2}\sigma^2 X(s)ds| = \mathcal{O}(\varepsilon^{1-\kappa_0-\gamma}). \tag{4.32}$$

Proof. By using Itô's formula we find

$$d(XZ_\varepsilon^2) = (dX)Z_\varepsilon^2 + X(dZ_\varepsilon^2) + (dX)(dZ_\varepsilon^2)$$

and

$$d(Z_\varepsilon^2) = 2(dZ_\varepsilon)Z_\varepsilon + (dZ_\varepsilon)^2 = 2Z_\varepsilon(-\varepsilon^{-2}Z_\varepsilon dT + \varepsilon^{-1}\sigma d\tilde{\beta}) + \varepsilon^{-2}\sigma^2 dT.$$

This gives

$$d(XZ_\varepsilon^2) = fZ_\varepsilon dT + gZ_\varepsilon d\tilde{\beta} - \varepsilon^{-2}2XZ_\varepsilon^2 dT + \varepsilon^{-1}2\sigma XZ_\varepsilon d\tilde{\beta} + \varepsilon^{-2}\sigma^2 X dT + \varepsilon^{-1}\sigma g dT.$$

We already know from the proof of Lemma 11 that $\sup_{T\in[0,T_0]}|Z_\varepsilon(T)| = \mathcal{O}(\varepsilon^{-\kappa_0})$ and it follows from Lemma 12 that $\sup_{T\in[0,\tau]}|X(T)| = \mathcal{O}(\varepsilon^{-\gamma})$. Therefore $d(XZ_\varepsilon^2)$ can be written as

$$d(XZ_\varepsilon^2) = -\varepsilon^{-2}2XZ_\varepsilon^2 dT + \varepsilon^{-2}\sigma^2 X dT + R_7 dT + R_8 d\tilde{\beta},$$

where $R_7(T)$ and $R_8(T)$ are stochastic processes with

$$\sup_{T\in[0,\tau]}|R_7(T)| = \mathcal{O}(\varepsilon^{-1-\kappa_0-\gamma}), \qquad \sup_{T\in[0,\tau]}|R_8(T)| = \mathcal{O}(\varepsilon^{-1-\kappa_0-\gamma}).$$

By multiplying with ε^2 and integrating from 0 to T it follows that

$$\int_0^T \tfrac{1}{2}\sigma^2 X ds - \int_0^T X Z_\varepsilon^2 ds = \tfrac{1}{2}\varepsilon^2 X Z_\varepsilon^2\Big|_0^T - \varepsilon^2 \int_0^T R_7 ds - \varepsilon^2 \int_0^T R_8 d\tilde{\beta}$$

and the application of Lemma 12 yields the desired result. □

4.3.5 SDE Lemma

With Lemma 13 we have closed the gap between the SDEs (1.5) and (4.17) down to some error on the right side which is of order $\varepsilon^{1-8\kappa_0}$. But to be able to compare the first Fourier mode a and the solution of the amplitude equation A we need the following Lemma.

Lemma 14. *Let $X_1(t), X_2(t) \in \mathbb{C}$ be stochastic processes given by*

$$\begin{aligned} X_1(t) &= X_1(0) + \int_0^t f(X_1) ds + \int_0^t g(X_1) d\beta \\ X_2(t) &= X_1(0) + \int_0^t f(X_2) ds + \int_0^t g(X_2) d\beta + R(t) \end{aligned} \tag{4.33}$$

with $sup_{t\in[0,\tau_0]}|R(t)| = \mathcal{O}(\varepsilon^\gamma)$, where $\gamma \in \mathbb{R}$ and $\tau_0 \leq T_0$ is a stopping time. Let there be a constant $C > 0$ and a process $\breve{R}(t)$ with $\sup_{t\in[0,\tau_0]}|\breve{R}(t)| = \mathcal{O}(\varepsilon^\gamma)$ such that the functions f and g satisfy the following conditions:

$$\mathrm{Re}\left\{(f(X_1) - f(X_2))\overline{\varphi}\right\} \leq C(|\varphi|^2 + |\breve{R}(t)|^2) \tag{4.34}$$

$$\forall x, y \in \mathbb{C} : \quad |g(x) - g(y)|^2 \leq C|x-y|^2, \tag{4.35}$$

where $\varphi := X_1 - (X_2 - R)$. Then the difference between X_1 and X_2 can be bounded by

$$\sup_{t\in[0,\tau_0]} |X_1(t) - X_2(t)| = \mathcal{O}(\varepsilon^\gamma). \tag{4.36}$$

Note that condition (4.34) can be established by a bound of the type

$$\mathrm{Re}\{(f(x) - f(y))(x - y - z)\} \leq C|x - y - z|^2 + p(y, z)$$

with polynomial p provided we have additional bounds on the process X_2.

Proof. Because of the unknown derivative of R it is much easier to split $X_1 - X_2$ into

$$X_1 - X_2 = \varphi - R \tag{4.37}$$

and bound $|\varphi|$ rather than the actual term.

Due to the stopping time the process φ is not easily bounded directly. Thus we extend all processes to $[0, T_0]$ and define

$$\tilde{R}(t) := \begin{cases} R(t) & \text{for } t \leq \tau_0 \\ R(\tau_0) & \text{for } t > \tau_0 \end{cases}$$

and modify X_1 and X_2:

$$\begin{aligned} \tilde{X_1}(t) &:= X_1(0) + \int_0^{\tau_0 \wedge t} f(\tilde{X}_1) ds + \int_0^t g(\tilde{X}_1) d\beta \\ \tilde{X_2}(t) &:= X_1(0) + \int_0^{\tau_0 \wedge t} f(\tilde{X}_2) ds + \int_0^t g(\tilde{X}_2) d\beta + \tilde{R}(t). \end{aligned}$$

With this we can define a suitable replacement for φ:

$$\begin{aligned} \varphi_{\tau_0}(t) &:= \tilde{X}_1(t) - (\tilde{X}_2(t) - \tilde{R}(t)) \\ &= \int_0^{\tau_0 \wedge t} (f(X_1) - f(X_2)) ds + \int_0^{\tau_0 \wedge t} (g(\tilde{X}_1) - g(\tilde{X}_2)) d\beta. \end{aligned}$$

Note that $\sup_{t \in [0, T_0]} |\tilde{R}(t)| = \mathcal{O}(\varepsilon^\gamma)$ and for any stopping time $\tau \leq \tau_0$ we have $\varphi_{\tau_0}(\tau) = \varphi(\tau)$, $\tilde{X}_1(\tau) = X_1(\tau)$ and $\tilde{X}_2(\tau) = X_2(\tau)$. This means

$$\sup_{t \in [0, \tau_0]} |\varphi(t)| = \sup_{t \in [0, \tau_0]} |\varphi_{\tau_0}(t)|.$$

Now in order to bound the moments of $\sup_{t \in [0, \tau_0]} |\varphi_{\tau_0}|$ we first need a bound on the moments of $|\varphi_{\tau_0}|$. We start by taking the differential of $|\varphi_{\tau_0}|^{2p}$ for $p \in \mathbb{N}$:

$$\begin{aligned} d|\varphi_{\tau_0}|^{2p} &= d(\overline{\varphi_{\tau_0}} \varphi_{\tau_0})^p = p(\overline{\varphi_{\tau_0}} \varphi_{\tau_0})^{p-1} d(\overline{\varphi_{\tau_0}} \varphi_{\tau_0}) \\ &= p|\varphi_{\tau_0}|^{2p-2} \left((d\overline{\varphi_{\tau_0}}) \varphi_{\tau_0} + \overline{\varphi_{\tau_0}} (d\varphi_{\tau_0}) + (d\overline{\varphi_{\tau_0}})(d\varphi_{\tau_0}) \right). \end{aligned}$$

The derivative of φ_{τ_0} is given by

$$d\varphi_{\tau_0} = \chi_{[0, \tau_0 \wedge t]} (f(X_1) - f(X_2)) dt + (g(\tilde{X}_1) - g(\tilde{X}_2)) d\beta \; .$$

Therefore

$$\begin{aligned} d|\varphi_{\tau_0}|^{2p} = p|\varphi_{\tau_0}|^{2p-2} [&\chi_{[0, \tau_0 \wedge t]} 2 \operatorname{Re} \{ \overline{\varphi_{\tau_0}} (f(X_1) - f(X_2)) \} \, dt \\ &+ 2 \operatorname{Re} \left\{ \overline{\varphi_{\tau_0}} (g(\tilde{X}_1) - g(\tilde{X}_2)) \right\} d\beta + |g(\tilde{X}_1) - g(\tilde{X}_2)|^2 dt]. \end{aligned}$$

Next we integrate and split the right-hand side into three parts:

$$\begin{aligned} |\varphi_{\tau_0}(t)|^{2p} = &\int_0^{\tau_0 \wedge t} p|\varphi_{\tau_0}|^{2p-2} 2 \operatorname{Re} \{ \overline{\varphi_{\tau_0}} (f(X_1) - f(X_2)) \} \, ds \\ &+ \int_0^t p|\varphi_{\tau_0}|^{2p-2} 2 \operatorname{Re} \left\{ \overline{\varphi_{\tau_0}} (g(\tilde{X}_1) - g(\tilde{X}_2)) \right\} d\beta \\ &+ \int_0^t p|\varphi_{\tau_0}|^{2p-2} |g(\tilde{X}_1) - g(\tilde{X}_2)|^2 ds \\ := &\, I_1 + I_2 + I_3. \end{aligned}$$

For the first part we can exchange φ and φ_{τ_0} freely because the integral reaches only up to the stopping time τ_0. Doing this and using (4.34) we find

$$\begin{aligned} I_1 &= \int_0^{\tau_0 \wedge t} p|\varphi|^{2p-2} 2\,\mathrm{Re}\,\{\overline{\varphi}(f(X_1) - f(X_2))\}\, ds \\ &\leq \int_0^{\tau_0 \wedge t} p|\varphi|^{2p-2} 2C(|\varphi|^2 + |\breve{R}|^2) ds \\ &\leq \int_0^{\tau_0 \wedge t} C_p(|\varphi_{\tau_0}|^{2p} + |\breve{R}|^{2p}) ds \leq C_p(\int_0^t (|\varphi_{\tau_0}|^{2p} ds + \int_0^{\tau_0} |\breve{R}|^{2p}) ds, \end{aligned}$$

where C_p is a constant depending on p and we used Young's inequality in the last step. The third part can be bounded from above by using (4.35) and a simple application of the triangle inequality:

$$\begin{aligned} I_3 &\leq \int_0^t p|\varphi_{\tau_0}|^{2p-2} |\tilde{X}_1 - \tilde{X}_2|^2 ds \\ &\leq \int_0^t p|\varphi_{\tau_0}|^{2p-2} (|\varphi_{\tau_0}|^2 + |\tilde{R}|^2) ds \leq \int_0^t C_p(|\varphi_{\tau_0}|^{2p} + |\tilde{R}|^{2p}) ds. \end{aligned}$$

Again we used Young's inequality in the last step. Now since stochastic integration preserves the local martingale property, taking the expectation value of $|\varphi_{\tau_0}|^{2p}$ yields, for all $t \leq T_0$,

$$\begin{aligned} \mathbb{E}(|\varphi_{\tau_0}(t)|^{2p}) &= \mathbb{E}(I_1) + \mathbb{E}(I_2) \\ &\leq C_p \mathbb{E}\Big(\int_0^t |\varphi_{\tau_0}|^{2p} + |\tilde{R}|^{2p} ds + \int_0^{\tau_0} |\breve{R}|^{2p} ds \Big) \\ &\leq \int_0^t C_p \mathbb{E}(|\varphi_{\tau_0}|^{2p}) ds + C_p T_0 R_{\mathrm{sup}}^{2p}, \end{aligned}$$

where $R_{sup}^{2p} := \mathbb{E}(\sup_{t\in[0,\tau_0]} |\breve{R}(t)|^{2p} + \sup_{t\in[0,T_0]} |\tilde{R}(t)|^{2p})$. Since $\sup_{t\in[0,\tau_0]} |\breve{R}(t)| = \mathcal{O}(\varepsilon^\gamma)$ and $\sup_{t\in[0,\tau_0]} |\tilde{R}(t)| = \mathcal{O}(\varepsilon^\gamma)$ we have

$$|R_{sup}^{2p}| = \mathcal{O}(\varepsilon^\gamma).$$

We apply Gronwalls Lemma to get

$$\begin{aligned} \mathbb{E}(|\varphi_{\tau_0}(t)|^{2p}) &\leq C_p T_0 R_{sup}^{2p} + \int_0^t C_p^2 T_0 R_{sup}^{2p} e^{(T_0 - s)C_p} ds \\ &\leq C_p T_0 R_{sup}^{2p} + C_p^2 T_0^2 R_{sup}^{2p} e^{T_0 C_p}. \end{aligned} \tag{4.38}$$

With this we can now bound the moments of $\sup_{t\in[0,\tau_0]} |\varphi_{\tau_0}(t)|$. We start with

$\mathbb{E}(\sup_{t\in[0,\tau_0]} I_3(t))$:

$$\begin{aligned}\mathbb{E}(\sup_{t\in[0,\tau_0]} I_3(t)) &= \mathbb{E}\sup_{t\in[0,\tau_0]}\Big(\int_0^t 2\,\mathrm{Re}\left\{\overline{\varphi_{\tau_0}}(g(\tilde{X}_1)-g(\tilde{X}_2))\right\}d\beta\Big)\\ &\le \mathbb{E}\Big(\int_0^{\tau_0} C_p^2|\varphi_{\tau_0}|^{4p-2}|g(\tilde{X}_1)-g(\tilde{X}_2)|^2ds\Big)^{1/2}\\ &\le \Big(\mathbb{E}\int_0^{T_0} C_p^2|\varphi_{\tau_0}|^{4p-2}(|\varphi_{\tau_0}|^2+|\tilde{R}|^2)ds\Big)^{1/2}\\ &\le C_p\Big(\mathbb{E}\int_0^{T_0}|\varphi_{\tau_0}|^{4p}+|\tilde{R}|^{4p}ds\Big)^{1/2},\end{aligned}$$

where we used the Burkholder Davis Gundy theorem in the second step, the Hölder inequality in the third and Young's inequality in the last step.

The whole term is now easily bounded by

$$\begin{aligned}\mathbb{E}(\sup_{t\in[0,\tau_0]}|\varphi(t)|)^{2p} &= \mathbb{E}(\sup_{t\in[0,\tau_0]}(I_1+I_2+I_3))\\ &\le C_p\mathbb{E}\Big(\int_0^{T_0}(|\varphi_{\tau_0}|^{2p}+|\tilde{R}|^{2p})ds+\int_0^{\tau_0}|\breve{R}|^{2p}ds\Big)\\ &\qquad + C_p\Big(\mathbb{E}\int_0^{T_0}|\varphi_{\tau_0}|^{4p}+|\tilde{R}|^{4p}ds\Big)^{1/2}\\ &\le C_p\Big(\int_0^{T_0}\mathbb{E}|\varphi_{\tau_0}|^{2p}ds\Big)+C_p\Big(\int_0^{T_0}\mathbb{E}|\varphi_{\tau_0}|^{4p}ds\Big)^{1/2}\\ &\qquad + C_p(T_0+T_0^{1/2})R_{\sup}^{2p}.\end{aligned}$$

Using (4.38) we attain

$$\begin{aligned}\mathbb{E}(\sup_{t\in[0,\tau_0]}|\varphi_{\tau_0}(t)|)^{2p} &\le C_pT_0(C_pT_0R_{\sup}^{2p}+C_p^2T_0^2R_{\sup}^{2p}e^{T_0C_p})\\ &\qquad + C_pT_0^{1/2}(C_{2p}T_0R_{\sup}^{4p}+C_{2p}^2T_0^2R_{\sup}^{4p}e^{T_0C_{2p}})+C_pT_0^{3/2}R_{\sup}^{2p}.\end{aligned}$$

Finally any moment can be bounded by even moments through Hölder interpolation, which proves that $\sup_{t\in[0,\tau_0]}|\varphi(t)| = \sup_{t\in[0,\tau_0]}|\varphi_{\tau_0}(t)| = \mathcal{O}(\varepsilon^\gamma)$. By assumption we also have that $\sup_{t\in[0,\tau_0]}|R(t)| = \mathcal{O}(\varepsilon^\gamma)$, so the result follows from (4.37). □

From what we have proven up to now it is easily shown that Theorem 7 holds at least until the time τ^*, but we still need to show that τ^* is large enough. For this we prove bounds on moments of A which are a direct application of Lemma 14.

Corollary 15. *Let $A(t)$ be the solution to the amplitude equation* (1.5) *from Theorem 7, then the following holds:*

$$\sup_{t\in[0,T_0]}|A(t)| = \mathcal{O}(\varepsilon^{-2\kappa}). \tag{4.39}$$

Note that $A(0)$ was defined in Theorem 7 such that $A(0) = \mathcal{O}(\varepsilon^{-\kappa})$.

Proof. We define f, g and R by

$$\begin{aligned} R(t) &:= -A(0) \\ f(A) &:= \nu A + 3(\tfrac{38}{27}\vartheta^2 - 1)A|A|^2 + 3(\vartheta^2 - \tfrac{1}{2})\sigma^2 A \\ g(A) &:= 2\sigma\vartheta A. \end{aligned} \tag{4.40}$$

With this we can write A and zero as in (4.33):

$$\begin{aligned} A(t) &= A(0) + \int_0^t f(A)dt + \int_0^t g(A)d\beta \\ 0 &= A(0) + \int_0^t f(0)dt + \int_0^t g(0)d\beta + R. \end{aligned}$$

Since $f(0) = g(0)$ we obtain $\sup_{t\in[0,T_0]} |R(t)| = \sup_{t\in[0,T_0]} |A(0)| = \mathcal{O}(\varepsilon^{-\kappa})$, and we derive the desired result directly from Lemma 14, provided we can prove the conditions (4.34) and (4.35). Because g is linear, (4.35) is readily verified:

$$|g(x) - g(y)|^2 = |2\sigma(x-y)|^2 \leq 4\sigma^2|x-y|^2. \tag{4.41}$$

This leaves (4.34). For better readability we write f as

$$f(X) = C_1 X - C_2|X|^2 X$$

with positive constants C_1 and C_2. For the linear part of f we are in the same position as for g, there is no dependency on X_1 or X_2:

$$\operatorname{Re}\{(\overline{X_1} - (\overline{X_2} - \overline{R}))(C_1 X_1 - X_2)\} \leq 3C_1(|X_1 - (X_2 - R)|^2 + |R|^2). \tag{4.42}$$

For the cubic term, to keep this proof simple, we note that it is sufficient to bound it here just for the special case $X_1 = A$ and $X_2 = 0$.

$$\begin{aligned} \operatorname{Re}\{(\overline{A} - (0 - \overline{R}))(-C_2|A|^2 A - 0)\} &= -C_2(|A|^4 + |A|^2 \operatorname{Re}\{\overline{R}A\}) \\ &\leq \begin{cases} 0 & \text{for } |A| > |R| \\ C_2(|R|^2)^2 & \text{for } |A| \leq |R| \end{cases}. \end{aligned}$$

With $\gamma = -2\kappa$ and setting $\breve{R} = R + R^2$ the assumptions for Lemma 14 are satisfied. ☐

4.3.6 Removing the error

Combining the lemmas of the previous sections, we are ready to prove Theorem 7.

Proof of Theorem 7. By Lemma 10 $u(t)$ can be approximated by $a = \hat{v}_1$ and Z_ε until the time τ^*:

$$\sup_{t\in[0,\varepsilon^{-2}\tau^*]} \|u(t) - \varepsilon a(\varepsilon^2 t)e^{ix} - \varepsilon\overline{a}(\varepsilon^2 t)e^{-ix} - \varepsilon Z_\varepsilon - e^{T\varepsilon^{-2}(1+\partial_x^2)^2} v_s(0)\|_\infty = \mathcal{O}(\varepsilon^{2-8\kappa_0}). \tag{4.43}$$

Now we bound the difference between a and A until time τ^*. The initial condition $A(0)$ is exactly the coefficient of the first Fourier mode of $v(0,x)$. This means $A(0) = a(0)$, thus by Lemma 11 and Lemma 12 we know that a is given by

$$a(t) = A(0) + \int_0^t (\nu a + 3(\tfrac{38}{27}\vartheta^2 - 1)a|a|^2 + 6(\vartheta^2 - \tfrac{1}{2})aZ_\varepsilon^2)ds + \int_0^t 2\sigma a d\tilde{\beta} + R_9,$$

where $\sup_{[0,\tau^*]} |R_9| = \mathcal{O}(\varepsilon^{1-8\kappa_0})$. Next we split the aZ_ε^2 term into

$$aZ_\varepsilon^2 = (a - R_9)Z_\varepsilon^2 + R_9 Z_\varepsilon^2.$$

The second part is bounded by $\sup_{[0,\tau^*]} |R_9 Z_\varepsilon^2| = \mathcal{O}(\varepsilon^{1-10\kappa_0})$ and the first part can be exchanged by using Lemma 13. Set as indicated after (4.6) $\kappa_0 = \frac{17}{8}\kappa$. Because

$$\sup_{[0,\tau^*]} |\nu a + 3(\frac{38}{27}\vartheta^2 - 1)a|a|^2 + 6(\vartheta^2 - \tfrac{1}{2})aZ_\varepsilon^2| = \mathcal{O}(\varepsilon^{-6\kappa_0}) \tag{4.44}$$

$$\sup_{[0,\tau^*]} |2\sigma a| = \mathcal{O}(\varepsilon^{-6\kappa_0}) \tag{4.45}$$

and $10\kappa_0 = \frac{85}{4}\kappa \leq 22\kappa$ we get

$$a(t) = A(0) + \int_0^t (\nu a + 3(\tfrac{38}{27}\vartheta^2 - 1)a|a|^2 + 3(\vartheta^2 - \tfrac{1}{2})\sigma^2 a)ds + \int_0^t 2\sigma a d\tilde{\beta} + R_{10},$$

where $\sup_{t\in[0,\tau^*]} |R_{10}(t)| = \mathcal{O}(\varepsilon^{1-22\kappa})$. With f and g defined as in (4.40) we show

that there exists a process $\breve{R}$ with

$$\sup_{t\in[0,\tau^*]} |\breve{R}(t)| = \mathcal{O}(\varepsilon^{1-28\kappa}) \tag{4.46}$$

such that the conditions (4.34) and (4.35) are fulfilled and we can apply Lemma 14. Since $\sup_{t\in[0,\tau^*]} |R_{10}| = \mathcal{O}(\varepsilon^{1-22\kappa})$ the condition on g and the linear term of f are already covered by (4.41) respectively (4.42). Because of this we only need show that there is a positive constant C and a process $\breve{R}$ conforming to (4.46) such that

$$\rho := \mathrm{Re}\left\{-C_2(\overline{A} - (\overline{a} - \overline{R_{10}}))(|A|^2 A - |a|^2 a)\right\} \leq C(|A - (a - R_{10})|^2 + |\breve{R}|^2),$$

where $C_2 = -3(\frac{38}{27}\vartheta^2 - 1)$ is a positive constant. We do this by splitting ρ into two parts:

$$\begin{aligned}
\rho &= \operatorname{Re}\left\{-C_2\left(\overline{A} - (\overline{a} - \overline{R_{10}})\right)\left(|A|^2A - |a|^2a\right)\right\} \\
&= \operatorname{Re}\left\{-C_2\left(\overline{A} - (\overline{a} - \overline{R_{10}})\right)\left(|A|^2A - |a - R_{10}|^2(a - R_{10})\right)\right\} \\
&\quad + \operatorname{Re}\left\{-C_2\left(\overline{A} - (\overline{a} - \overline{R_{10}})\right)\left(|a - R_{10}|^2(a - R_{10}) - |a|^2a\right)\right\} \\
&=: \rho_1 + \rho_2.
\end{aligned}$$

The first term is negative because for any two complex numbers z, w we have

$$\begin{aligned}
&2\operatorname{Re}\{(\overline{z} - \overline{w})(|z|^2z - |w|^2w\} \\
&\quad = 2|z-w|^2(|z|^2 + |w|^2) + 2\operatorname{Re}\{(z-w)^2\overline{zw}\} \\
&\quad \geq 2|z-w|^2(|z|^2 + |w|^2) - |z-w|^2(|z|^2 + |w|^2) \\
&\quad \geq |z-w|^2(|z|^2 + |w|^2) \geq 0.
\end{aligned}$$

This means ρ_1 can be bounded from above by 0. The second term can be bounded by

$$\begin{aligned}
|\rho_2| &\leq C_2|\overline{A} - (\overline{a} - \overline{R_{10}})|(3|a|^2|R_{10}| + 3|a||R_{10}|^2 + |R_{10}|^3) \\
&\leq C_2(|\overline{A} - (\overline{a} - \overline{R_{10}})|^2 + (3|a|^2|R_{10}| + 3|a||R_{10}|^2 + |R_{10}|^3)^2)
\end{aligned}$$

and since $\sup_{t\in[0,\tau^*]}|a(t)| = \mathcal{O}(\varepsilon^{-3\kappa})$ we obtain (as $\kappa < \frac{1}{27}$)

$$\sup_{t\in[0,\tau^*]}\left(3|a|^2|R_{10}| + 3|a||R_{10}|^2 + |R_{10}|^3\right) = \mathcal{O}(\varepsilon^{1-28\kappa}).$$

Therefore Lemma 14 yields the following bound on $|A-a|$:

$$\sup_{t\in[0,\tau^*]}|A(t) - a(t)| = \mathcal{O}(\varepsilon^{1-28\kappa}). \tag{4.47}$$

Combining this with Corollary 15 we obtain

$$\sup_{t\in[0,\tau^*]}|a(t)| \leq \sup_{t\in[0,\tau^*]}|A(t) - a(t)| + \sup_{t\in[0,\tau^*]}|A(t)| = \mathcal{O}(\varepsilon^{-2\kappa}). \tag{4.48}$$

Next we show that the probability $\mathbb{P}(\tau^* < T_0)$ is small. Define the following subset of the probability space Ω:

$$M := \{\omega \in \Omega : \tau^*(\omega) < T_0\}.$$

If $\omega \in M$ then it follows from the definition of τ^* that $\|v(\tau^*(\omega))\|_\infty = \varepsilon^{-\kappa_0}$. Therefore the moments of $\|v(\tau^*)\|_\infty$ can be written as follows

$$\mathbb{E}\|v(\tau^*)\|_\infty^q = \int_{M^c}\|v(\tau^*)\|_\infty^q d\mathbb{P} + \int_M (\varepsilon^{-\kappa_0})^q d\mathbb{P} \geq \mathbb{P}(M)\varepsilon^{-q\kappa_0},$$

where $M^c := \Omega \setminus M$ is the complement set of M. From (4.48), (4.21),(4.13) and (4.12) we have

$$\begin{aligned}\mathbb{E}\|v(\tau^*)\|_\infty^q &\le C_q \mathbb{E} \sup_{t\in[0,\tau^*]} (|a(t)|^q + |Z_\varepsilon(t)|^q + |\hat{v}_0(t) - Z_\varepsilon(t) - e^{-\varepsilon^{-2}T}\hat{v}_0(0)|^q) \\ &\quad + C_q \mathbb{E} \sup_{t\in[0,\tau^*]} \|\sum_{k\ge 2} \hat{v}_k - e^{-\varepsilon^{-2}T(1-k^2)^2}\hat{v}_k(0))e^{ikx}\|_\infty^q \\ &\quad + C_q \mathbb{E} \sup_{t\in[0,\tau^*]} \|e^{-\varepsilon^{-2}T\mathcal{L}} \sum_{k\ne 1} (\hat{v}_k(0))e^{ikx}\|_\infty^q \\ &\le C_q \varepsilon^{-2p\kappa}\end{aligned}$$

with a constant C_q depending on q, where we used that there is a constant C such that for all $u \in L^2_{per}$,

$$\|e^{-\varepsilon^{-2}T\mathcal{L}}u\|_\infty \le C\|u\|_\infty,$$

as proven in Corollary A.3 (see Appendix A). Therefore for any positive number q the probability of M is bounded by

$$\mathbb{P}(M) \le C_q \varepsilon^{q(\kappa_0 - 2\kappa)}. \tag{4.49}$$

Define

$$\xi := u(t) - \varepsilon A(\varepsilon^2 t)e^{ix} - \varepsilon \bar{A}(\varepsilon^2 t)e^{-ix} + \varepsilon Z_\varepsilon(\varepsilon^2 t) - e^{-t(1+\partial_x^2)^2} u_s(0).$$

The last step is now to bound the probability of $\sup_{t\in[0,\varepsilon^{-2}T_0]}\|\xi\|_\infty$ being too large (i.e. $\mathbb{P}(\sup_{t\in[0,\varepsilon^{-2}T_0]}\|\xi\|_\infty > \varepsilon^{2-29\kappa})$). We can split this into

$$\begin{aligned}\mathbb{P}(\sup_{t\in[0,\varepsilon^{-2}T_0]} \|\xi\|_\infty > \varepsilon^{2-29\kappa}) &= \mathbb{P}(M \cap \{\sup_{t\in[0,\varepsilon^{-2}T_0]} \|\xi\|_\infty > \varepsilon^{2-29\kappa}\}) \\ &\quad + \mathbb{P}(M^c \cap \{\sup_{t\in[0,\varepsilon^{-2}T_0]} \|\xi\|_\infty > \varepsilon^{2-29\kappa}\}) \\ &=: P_1 + P_2.\end{aligned}$$

P_1 is easily bounded by

$$\mathbb{P}(M \cap \{\sup_{t\in[0,\varepsilon^{-2}T_0]} \|\xi\|_\infty > \varepsilon^{2-29\kappa})\}) \le \mathbb{P}(M) \le C_q \varepsilon^{q(\kappa_0-2\kappa)},$$

so the only thing left to do is to bound P_2. We get

$$P_2 = \mathbb{P}(M^c \cap \{\sup_{t\in[0,\varepsilon^{-2}T_0]} \|\xi\|_\infty > \varepsilon^{2-29\kappa}\}) \le \mathbb{P}(\sup_{t\in[0,\varepsilon^{-2}\tau^*]} \|\xi\|_\infty > \varepsilon^{2-29\kappa}).$$

From (4.43) and (4.47) it follows that (since $8\kappa_0 < 28\kappa$)

$$\sup_{t\in[0,\varepsilon^{-2}\tau^*]} \|\xi\|_\infty = \mathcal{O}(\varepsilon^{2-28\kappa}).$$

Thus using the Chebychev inequality yields

$$P_2 \le \frac{1}{\varepsilon^{q(2-29)\kappa}} \mathbb{E}(\sup_{t\in[0,\varepsilon^{-2}\tau^*]} \|\xi\|_\infty^q) \le C_q \varepsilon^{q\kappa}$$

and by choosing $q = \max\{\frac{p}{\kappa}, \frac{p}{\kappa_0 - 2\kappa}\}$ we get the desired result. □

5 Result for the unbounded domain

5.1 Approximation Theorem for the unbounded domain

To state our second main result we still need some additional definitions:

Definition 16. *For $u \in \mathcal{H}^\alpha \oplus \mathbb{R}$ we use the following projections*

$$P_c u := \operatorname*{ess.lim}_{x\to\infty} u(x) \qquad P_a u := \int_{\frac{3}{4}\varepsilon^{-1}}^{\frac{5}{4}\varepsilon^{-1}} \mathcal{F}(u - P_c u) e^{ikx} dk$$

$$P_+ u := \int_0^\infty \mathcal{F}(u - P_c u) e^{ikx} dk \qquad P_- u := \int_{-\infty}^0 \mathcal{F}(u - P_c u) e^{ikx} dk.$$

The projection P_c onto the constant is the same as defined in (2.1) of Chapter 2 but repeated here for better readability. With these definitions the result reads as follows:

Theorem 17. *Let T_0 be a time of order 1, $\frac{1}{2} < \alpha < 1$, $\vartheta \in \mathbb{R}$ with $\vartheta^2 \leq \frac{27}{38}$ and $0 < \kappa < \frac{1-\alpha}{5}$. Let u be a stochastic process with paths $u_\omega(t,x) \in C^0([0,T_0]; \{\mathcal{H}^\alpha \oplus \mathbb{R}\})$ that is a mild solution of the SPDE*

$$du = (-\mathcal{L}u + \nu\varepsilon^2 u + \vartheta u^2 - u^3)dt + \varepsilon\sigma d\beta, \tag{5.1}$$

where $\mathcal{L} := (1+\partial_x^2)^2$, $\beta(t)$ is a real valued standard Brownian motion, and the initial value fulfils the bound

$$\|(P_+ u(0,\varepsilon^{-1}x))e^{-i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} + |P_c u(0,x)| = \mathcal{O}(\varepsilon^{1-\kappa}). \tag{5.2}$$

Also let A be a stochastic process with paths $A_\omega(t) \in C([0,T_0]; L^\infty)$ that is a mild solution of the SPDE

$$\begin{aligned} dA &= (4\partial_x{}^2 A + \nu A + 3(\frac{38}{27}\vartheta^2 - 1)A|A|^2 + 3(\vartheta^2 - \frac{1}{2})\sigma^2 A)dt + 2\vartheta\sigma A d\tilde{\beta} \\ A(0) &= \varepsilon^{-1}(P_a u(0,\varepsilon^{-1}x))e^{-i\varepsilon^{-1}x}, \end{aligned} \tag{5.3}$$

where $\tilde{\beta}(T) := \varepsilon\beta(\varepsilon^{-2}T)$ is a rescaled version of $\beta(t)$.

Then, for all $p \in \mathbb{N}$, there is a constant C_p such that the following holds:

$$i)\ \mathbb{P}\Big(\sup_{t\in[0,\varepsilon^{-2}T_0]} \|P_+(u(t,\varepsilon^{-1}x) - \varepsilon A(\varepsilon^2 t, x)e^{i\varepsilon^{-1}x})e^{-i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} + \sup_{t\in[0,\varepsilon^{-2}T_0]} \|P_-(u(t,\varepsilon^{-1}x) - \varepsilon\overline{A(\varepsilon^2 t, x)}e^{-i\varepsilon^{-1}x})e^{i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} + \sup_{t\in[0,\varepsilon^{-2}T_0]} |P_c u(t) - \varepsilon Z_\varepsilon(\varepsilon^2 t) - P_c e^{-t}u(0)| > \varepsilon^{1+\alpha\wedge(1-\alpha)-26\kappa}\Big) \le C_p\varepsilon^p \tag{5.4}$$

$$ii)\ \mathbb{P}\Big(\sup_{t\in[0,\varepsilon^{-2}T_0]} \|u(t,x) - \varepsilon A(\varepsilon^2 t, \varepsilon x)e^{ix} - \varepsilon\overline{A(\varepsilon^2 t, \varepsilon x)}e^{-ix} - \varepsilon Z_\varepsilon(\varepsilon^2 t) - P_c e^{-t}u(0)\|_\infty > \varepsilon^{1+\alpha-26\kappa}\Big) \le C_p\varepsilon^p, \tag{5.5}$$

where Z_ε is as before the Ornstein-Uhlenbeck process defined by

$$Z_\varepsilon(T) := \sigma\int_0^T e^{-\varepsilon^{-2}(t-s)}d\tilde{\beta}(s). \tag{5.6}$$

The idea behind this approximation comes from looking at the bounded case and increasing the size of the domain: The Fourier series of $u \in L^2_{per}[0, 2n\pi]$ is given by

$$u(x) = n^{-1}\sum_{k\in\mathbb{N}}(\mathcal{F}u)(n^{-1}k)e^{in^{-1}kx}.$$

The dominant part (i.e the modes around the roots of the spectrum of the operator $\mathcal{L}$) can be written as

$$\begin{aligned} u_d &:= n^{-1}\sum_{|1+n^{-1}k|\le\delta}(\mathcal{F}u)(n^{-1}k)e^{in^{-1}kx} \\ &= n^{-1}\Big(\sum_{|n^{-1}k|\le\delta}(\mathcal{F}u)(1+n^{-1}k)e^{in^{-1}kx}\Big)\ e^{ix} \\ &\quad + n^{-1}\Big(\sum_{|n^{-1}k|\le\delta}(\mathcal{F}u)(-1+n^{-1}k)e^{in^{-1}kx}\Big)\ e^{-ix}\ , \end{aligned}$$

which when going to the limit $n \to \infty$ becomes the integral

$$u_d = \Big(\int_{-\delta}^{\delta}(\mathcal{F}u)(1+k)e^{ikx}dk\Big)e^{ix} + \Big(\int_{-\delta}^{\delta}(\mathcal{F}u)(-1+k)e^{ikx}dk\Big)e^{-ikx} =: u_d^+ + u_d^-.$$

Further the operator $\mathcal{L} = (1+\partial_x^2)^2$ can then, similar to the bounded case, be exchanged for a local approximation of its spectrum in a neighbourhood of ± 1. Here the neighbourhood does not consist of single points but the intervals $[\pm 1-\delta, \pm 1+\delta]$. On these intervals we have

$$
\begin{aligned}
&e^{-t\mathcal{L}}(u_d^+ + u_d^-) \\
&\quad = \Big(\int_{-\delta}^{\delta} e^{-t(1-(1+k)^2)^2}(\mathcal{F}u)(1+k)e^{ikx}dk\Big)e^{ix} \\
&\qquad\qquad + \Big(\int_{-\delta}^{\delta} e^{-t(1-(-1+k)^2)^2}(\mathcal{F}u)(-1+k)e^{ikx}dk\Big)e^{-ix} \\
&\quad \simeq \Big(\int_{-\delta}^{\delta} e^{-4tk^2}(\mathcal{F}u)(1+k)e^{ikx}dk\Big)e^{ix} + \Big(\int_{-\delta}^{\delta} e^{-4tk^2}(\mathcal{F}u)(-1+k)e^{ikx}dk\Big)e^{-ikx}
\end{aligned}
$$

and we could replace $-\mathcal{L}$ by $4\partial_x^2$. However since we want to rescale to the slow time $T = \varepsilon^2 t$ and keep the amplitude equation independent of ε we also have to rescale the space variable to the "slow" space $X = \varepsilon x$ which leads to the approximation

$$
\begin{aligned}
e^{-t\mathcal{L}}u_d^+(x,t) &\simeq \int_{-\delta}^{\delta} e^{-\varepsilon^{-2}4Tk^2}\varepsilon^{-1}(\mathcal{F}_X u(\varepsilon^{-1}X, \varepsilon^{-2}T))(\varepsilon^{-1} + \varepsilon^{-1}k)e^{ik\varepsilon^{-1}X}dk)e^{ix} \\
&= \int_{-\delta\varepsilon^{-1}}^{\delta\varepsilon^{-1}} e^{-4Tk^2}(\mathcal{F}_X u(\varepsilon^{-1}X, \varepsilon^{-2}T))(\varepsilon^{-1} + k)e^{ikX}dk)e^{ix},
\end{aligned}
$$

where $\mathcal{F}_X$ means the Fourier transformation with respect to X. (The term u_d^- is approximated exactly in the same way.) So for the rescaled solution $u(\varepsilon^{-1}X, \varepsilon^{-2}T)$ the operator can be exchanged with $4\partial_X^2$. The detailed proof is found in Section 5.2.6. As a result of the spatial rescaling the solution has the form of a modulated wave as pictured in Figure 5.1.

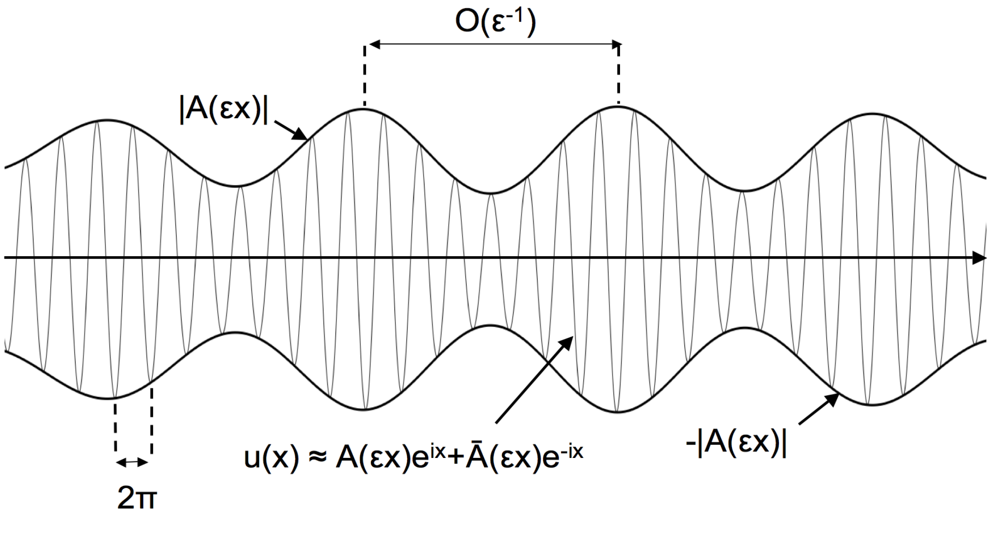

Figure 5.1 The solution as a modulated wave

The assumption on the initial condition translates to the Fourier transform $\mathcal{F}(u_0(x))$ of the initial value $u_0 := (1 - P_c)u(0)$ having a sharp peak of about order $\mathcal{O}(\varepsilon^{1/2})$

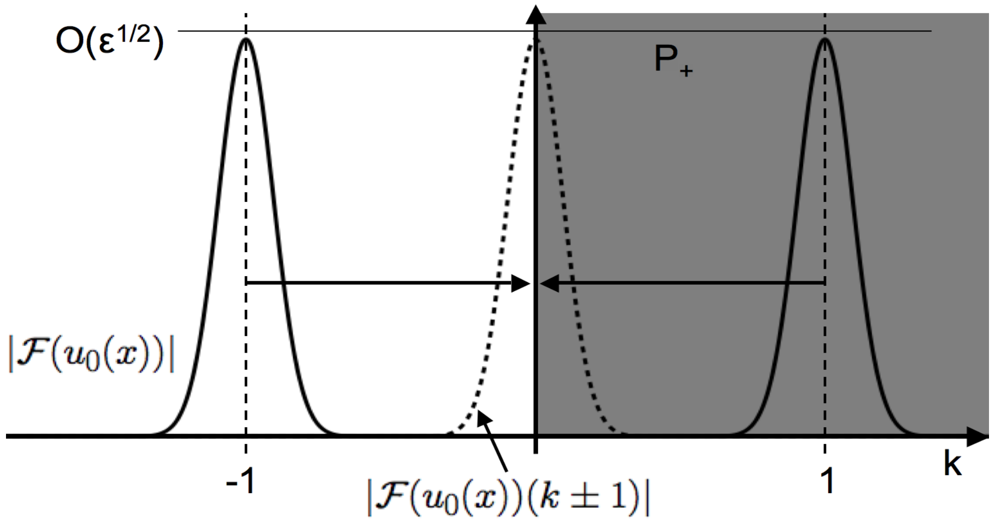

Figure 5.2 Fourier transform of the initial value

around $k = \pm 1$, as illustrated in Figure 5.2. (The solution u is real valued so the absolute value of the Fourier transform is symmetric around 0.) This can be seen as follows. Assumption (5.2) states that the projection of $\mathcal{F}(u_0(\varepsilon^{-1}x))$ onto the positive frequencies, shifted to the left by ε^{-1} has a $\mathcal{H}^\alpha$-norm of order $\mathcal{O}(\varepsilon^{1-\kappa})$ i.e.

$$\left(\int_{-\varepsilon^{-1}}^{\infty} (1+k^2)^\alpha |\mathcal{F}(u_0(\varepsilon^{-1}x))|^2 (k+\varepsilon^{-1}) dk \right)^{1/2} = \mathcal{O}(\varepsilon^{1-\kappa}).$$

Since $\mathcal{F}(u_0(\varepsilon^{-1}x))(k) = \varepsilon \mathcal{F}(u_0(x))(\varepsilon k)$ it follows by substitution that

$$\left(\int_{-1}^{\infty} (1+\varepsilon^{-2}k^2)^\alpha |\mathcal{F}(u_0(x))(k+1)|^2 dk \right)^{1/2} = \mathcal{O}(\varepsilon^{1/2-\kappa}).$$

Alternatively, we can state the initial condition as the rescaled initial value $v_0(x) = \varepsilon^{-1}u_0(\varepsilon^{-1}x)$ having “normal” peaks of about order $\mathcal{O}(1)$ around $k = \pm\varepsilon^{-1}$ or

$$\|(P_+ v_0(x))e^{-i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} = \mathcal{O}(\varepsilon^{-\kappa}).$$

The approximation result (5.4) can be interpreted in the same way.

The assumption on the initial condition where we already require a concentration of the “mass” of the Fourier transform in the vicinity of ± 1 instead of a simple bound on the supremum norm is one of the main differences to the bounded case and to Proposition 6. It is needed due to the fact that we can not control the Fourier transform through the supremum norm as we could for the bounded case.
Because of this we would, by means of the supremum norm, have no control on the coupling terms that arise from the quadratic nonlinearity and map back to the frequencies around ± 1. Especially we can not do estimates of the form

$$|(\widehat{v^n})_k| \le \|v^n\|_{L^2} \le \sqrt{2\pi}\|v^n\|_\infty = \sqrt{2\pi}\|v\|_\infty^n.$$

which we used to bound the Fourier modes of powers of u in the bounded domain case.

The same reason leads to another issue which is the reason for the restriction $\alpha \geq \frac{1}{2}$. As we bound $\mathcal{F}(u^2)$ and $\mathcal{F}(u^3)$ through estimates of the type (see Lemma 18)

$$\|u^n\|_{\mathcal{H}^\alpha} \leq \|u\|_\infty^{n-1}\|u\|_{\mathcal{H}^\alpha} \leq \|u\|_{\mathcal{H}^\alpha}^n,$$

we use the embedding $\mathcal{H}^\alpha \hookrightarrow L^\infty$ which is only valid for $\alpha > \frac{1}{2}$.

5.2 Proof of the result

5.2.1 Preliminaries

We start with some technical lemmas and definitions that are used throughout the main part of the proof. First we want to control the $\mathcal{H}^\alpha$ norm of products of $\mathcal{H}^\alpha$-functions. Powers can be bound by a Theorem from [RS96]:

Lemma 18. *Let $0 \leq \alpha < \mu$ and $u \in \mathcal{H}^\alpha$. Then*

$$\|u^\mu\|_{\mathcal{H}^\alpha} \leq C\|u\|_\infty^{\mu-1}\|u\|_{\mathcal{H}^\alpha}, \tag{5.7}$$

where C is a positive constant.

Proof. Special case of Theorem 1 in Section 5.4.3 on p.363 in [RS96]. □

If we want to bound products we can do this for $\alpha \leq 1$ by the following Lemma.

Lemma 19. *Let $u, v \in \mathcal{H}^\alpha \cap L^\infty$ with $0 < \alpha \leq 1$, then there is a constant $C > 0$ such that the following holds:*

$$\|uv\|_{\mathcal{H}^\alpha} \leq C(\|u\|_\infty\|v\|_{\mathcal{H}^\alpha} + \|v\|_\infty\|u\|_{\mathcal{H}^\alpha}). \tag{5.8}$$

In particular uv is also a function in $\mathcal{H}^\alpha$.

Proof. Let $\alpha < 1$. For the operator

$$D^\alpha u := \left(\int_{\mathbb{R}} \frac{|u(x+y) - P_x^{\lfloor\alpha\rfloor}u(x+y)|^2}{|y|^{2\alpha+1}}dy\right)^{1/2}$$

with $P_x^{\lfloor\alpha\rfloor}u$ being the first $\lfloor\alpha\rfloor$ summands of the Taylor series of u around x, there exists a constant $C > 0$ such that

$$C^{-1}\|u\|_{\mathcal{H}^\alpha} \leq \|u\|_{L^2} + \|D^\alpha u\|_{L^2} \leq C\|u\|_{\mathcal{H}^\alpha}.$$

For a proof see for example [AH96]. Therefore it is enough to show that

$$\|uv\|_{L^2} + \|D^\alpha uv\|_{L^2} \leq \|u\|_\infty(\|v\|_{L^2} + \|D^\alpha v\|_{L^2}) + \|v\|_\infty(\|u\|_{L^2} + \|D^\alpha u\|_{L^2}).$$

Obviously we have

$$\|uv\|_{L^2} \leq \|u\|_\infty \|v\|_{L^2} + \|v\|_\infty \|u\|_{L^2}.$$

Now

$$P_x^{\lfloor\alpha\rfloor} u(x+y) = u(x).$$

So we derive

$$\begin{aligned}
\|D^\alpha uv\|_{L^2}^2 &= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|(uv)(x+y) - P_x^{\lfloor\alpha\rfloor}(uv)(x+y)|^2}{|y|^{2\alpha+1}} dydx \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|u(x+y)(v(x+y)-v(x)) + v(x)(u(x+y)-u(x))|^2}{|y|^{2\alpha+1}} dydx \\
&\leq \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\|u\|_\infty^2 |v(x+y)-v(x)|^2 + \|v\|_\infty^2 |u(x+y)-u(x)|^2}{|y|^{2\alpha+1}} dydx \\
&= \|u\|_\infty^2 \|D^\alpha v\|_{L^2}^2 + \|v\|_\infty^2 \|D^\alpha u\|_{L^2}^2.
\end{aligned}$$

In the case $\alpha = 1$ we obtain

$$u \in \mathcal{H}^1 \cap L^\infty \Leftrightarrow \|u\| + \|Du\| < \infty$$

and therefore

$$\|D(uv)\| = \|u(Dv) + (Du)v\| \leq \|u\|_\infty \|Dv\|_{L^2} + \|v\|_\infty \|Du\|_{L^2}.$$

□

This will generally be used in combination with the embedding

$$\|u\|_\infty \leq (\int_{\mathbb{R}} (1+k^2)^{-\alpha} dk)^{1/2} (\int_{\mathbb{R}} (1+k^2)^{-\alpha} |\mathcal{F}(u)|^2 dk)^{1/2} \leq C\|u\|_{\mathcal{H}^\alpha}, \qquad (5.9)$$

which is valid for $\alpha > 1/2$.

As we rescale and shift functions in Fourier space we need the following operators.

Definition 20. *Let $u \in \mathcal{H}^\alpha \oplus \mathbb{R}$ i.e. $u = v + m$ with $v \in \mathcal{H}^\alpha$ and $m \in \mathbb{R}$. Define the operators $\mathcal{L}_\varepsilon^{+n}$, $n \in \mathbb{Z}$ by*

$$\begin{aligned}
\mathcal{L}_\varepsilon u &:= (1 + \varepsilon^2 \partial_x^2)^2 u = \int_{\mathbb{R}} (1 - \varepsilon^2 k^2)^2 \mathcal{F}(v)\, e^{ikx}\, dk - m \\
\mathcal{L}_\varepsilon^{+n} u &:= (1 + (\varepsilon\partial_x + in)^2)^2 u = \int_{\mathbb{R}} (1 - (\varepsilon k + n)^2)^2 \mathcal{F}(v)\, e^{ikx}\, dk - (1-n^2)^2 m.
\end{aligned}$$

Remark 21. *The operators $\mathcal{L}_\varepsilon$ and $\mathcal{L}_\varepsilon^{+n}$ are related in the following way. For $u \in \mathcal{H}^\alpha$ we have:*

$$\begin{aligned}\mathcal{L}_\varepsilon(ue^{in\varepsilon^{-1}x}) &= \int_{\mathbb{R}} (1-(\varepsilon k)^2)^2 \mathcal{F}(v)(k-n\varepsilon^{-1})\, e^{ikx}\, dk \\ &= \int_{\mathbb{R}} (1-(\varepsilon k+n)^2)^2 \mathcal{F}(v)\, e^{ikx}\, dk\, e^{in\varepsilon^{-1}} \\ &= (\mathcal{L}_\varepsilon^{+n} u)e^{in\varepsilon^{-1}x}.\end{aligned} \tag{5.10}$$

The semigroups generated by $\mathcal{L}$ and $\mathcal{L}_\varepsilon^{+n}$ are bounded on L^∞. For a proof of the following see Appendix A.

Corollary A.2. *The semigroups $e^{-t\mathcal{L}} := S_{\mathbb{R}}(t)$ defined in (2.2) and $e^{-t\mathcal{L}_\varepsilon^{+n}}$ defined through Definition 20 are bounded on $L^\infty(\mathbb{R})$ for all $t \geq 0$:*

$$\begin{aligned}\|e^{-t\mathcal{L}} f\|_\infty &\leq C\|f\|_\infty \\ \|e^{-t\mathcal{L}_\varepsilon^{+n}} f\|_\infty &\leq (C+n^4)\|f\|_\infty,\end{aligned}$$

where $C > 0$ is a constant independent of t.

We end this section by collecting here all projectors we use in the proof later. The following projectors were already defined.

$$P_c u := \operatorname*{ess.\,lim}_{x\to\infty} u(x) \qquad\qquad P_+ u := \int_0^\infty \mathcal{F}(u-P_c u)e^{ikx}dk$$

$$P_- u := \int_{-\infty}^0 \mathcal{F}(u-P_c u)e^{ikx}dk \qquad\qquad P_a u := \int_{\frac{3}{4}\varepsilon^{-1}}^{\frac{5}{4}\varepsilon^{-1}} \mathcal{F}(u-P_c u)e^{ikx}dk$$

Additionally, we need

Definition 22. *For $u \in \mathcal{H}^\alpha \oplus \mathbb{R}$ define*

$$P_\Psi u := \int_{-\frac{1}{4}\varepsilon^{-1}}^{1/4\varepsilon^{-1}} \mathcal{F}(u-P_c u)e^{ikx}dk, \qquad P_\Phi u := \int_{\frac{7}{4}\varepsilon^{-1}}^{\frac{9}{4}\varepsilon^{-1}} \mathcal{F}(u-P_c u)e^{ikx}dk$$

and the projections onto the sets

$$\begin{aligned} S &:= (-\infty, -\tfrac{9}{8}\varepsilon^{-1}) \cup (-\tfrac{7}{8}\varepsilon^{-1}, \tfrac{7}{8}\varepsilon^{-1}) \cup (\tfrac{9}{8}\varepsilon^{-1}, \infty) \\ S2 &:= (-\infty, -\tfrac{9}{4}\varepsilon^{-1}) \cup (-\tfrac{7}{4}\varepsilon^{-1}, -\tfrac{5}{4}\varepsilon^{-1}) \cup (-\tfrac{3}{4}\varepsilon^{-1}, -\tfrac{1}{4}\varepsilon^{-1}) \\ &\qquad \cup (\tfrac{1}{4}\varepsilon^{-1}, \tfrac{3}{4}\varepsilon^{-1}) \cup (\tfrac{5}{4}\varepsilon^{-1}, \tfrac{7}{4}\varepsilon^{-1}) \cup (\tfrac{9}{4}\varepsilon^{-1}, \infty),\end{aligned}$$

i. e.

$$P_s u := \int_S \mathcal{F}(u-P_c u)e^{ikx}dk \qquad \text{and} \quad P_{s2} u := \int_{S2} \mathcal{F}(u-P_c u)e^{ikx}dk.$$

Also define

$$P_s^c u := (1-P_s)u.$$

Figure 5.3 illustrates the frequency bands which these maps project onto. We further discuss the ideas behind these projectors in the next sections where we take a closer look at the spatial structure of the solution.

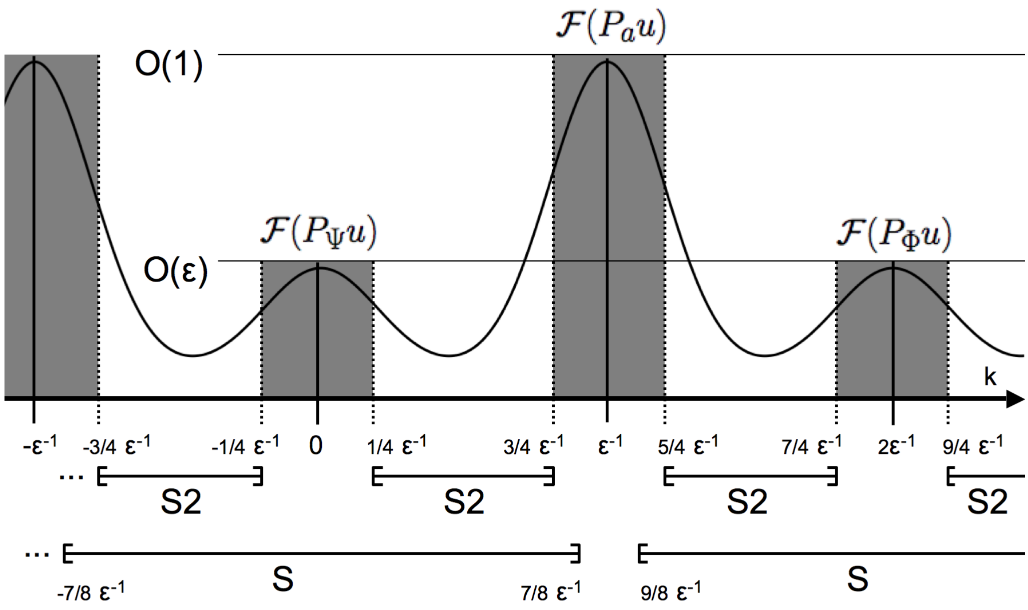

Figure 5.3 Projections onto the Fourier-spectrum

5.2.2 The rescaled solution

We rescale $u(t,x)$ as

$$v(T,X) := \varepsilon^{-1}u(\varepsilon^{-2}T, \varepsilon^{-1}X).$$

The stochastic differential equation is given by

$$dv = (-\varepsilon^{-2}\mathcal{L}_\varepsilon v + \nu v + \varepsilon^{-1}\vartheta v^2 - v^3)dT + \varepsilon^{-1}\sigma d\tilde{\beta}.$$

Because $u(t,x)$ is real valued, so is $v(T,X)$ and we have $v = \overline{v}$ from which follows that

$$\mathcal{F}(v)(-k) = \overline{\mathcal{F}(v)(k)},$$

which means $\|P_+ v(T,X)e^{-i\varepsilon^{-1}X}\|_{\mathcal{H}^\alpha} = \|P_- v(T,X)e^{i\varepsilon^{-1}X}\|_{\mathcal{H}^\alpha}$. Also we define

$$\kappa_0 := \tfrac{25}{24}\kappa > \kappa \tag{5.11}$$

and the stopping time

$$\tau^* = \inf\left\{T > 0 : \|P_+ v(T,X)e^{-i\varepsilon^{-1}X}\|_{\mathcal{H}^\alpha} + |P_c v| > \varepsilon^{-\kappa_0}\right\} \wedge T_0. \tag{5.12}$$

Up until this stopping time the Fourier transform of $v(T,X)$ has, by definition of τ^*, peaks of about order one at $\mathcal{F}(v)(\pm\varepsilon^{-1})$. This resembles the dominant first modes in the bounded case. Translating the interaction structure from Figure 4.2 to this new setup we should see additional peaks of about order $\mathcal{O}(\varepsilon)$ at the frequencies $\pm 2\varepsilon^{-1}$ and 0 as illustrated in Figure 5.3. This is indeed the case as will be proved in the next section in Lemma 5.13.
The smaller peaks are contained in the projection $P_s v$ onto the set S which excludes the frequencies around $\pm\varepsilon^{-1}$. The set $S2$ excludes all mentioned peaks and is therefore supposed to contain only frequencies that get pushed down by the semigroup to an order of about $\mathcal{O}(\varepsilon^2)$. We also show this in Lemma 5.13.

5.2.3 Decomposition of the rescaled solution

In order to get control of the main peaks in the form of their Itô differentials, it is essential (as we will see later) that we leave a spectral gap between them. These gaps occurred naturally in the bounded case as the Fourier series was discrete, but here they (i.e. the set $S2$) have to be handled separately.
Also we need to show that the non-dominant peaks are small and the constant "mode" $P_c v$ can be approximated by the fast OU-process Z_ε which is the solution of

$$dZ_\varepsilon = -\varepsilon^{-2} Z_\varepsilon dT + \sigma\varepsilon^{-1} d\tilde{\beta}, \quad Z_\varepsilon(0) = 0. \tag{5.13}$$

This is done in the next Lemma:

Lemma 23. *Under the same assumptions as Theorem 17, with stopping time τ^* defined by* (5.12)*, there exist decompositions*

$$v = P_c v + \sum_{j=-3}^{3} v_j$$

$$v - e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon T} v(0) = P_c v + \sum_{j=-3}^{3} \tilde{v}_j$$

of v and $v - e^{-\varepsilon^{-2}T\mathcal{L}_\varepsilon} v(0)$ such that for all $-3 \le j \le 3$:

$$\sup_{T\in[0,\tau^*]} \|(P_s v_j(T)) e^{-ij\varepsilon^{-1}}\|_{\mathcal{H}^\alpha} = \mathcal{O}(\varepsilon^{1-4\kappa_0}) \tag{5.14}$$

$$\sup_{T\in[0,\tau^*]} \|(P_{s2} \tilde{v}_j(T)) e^{-ij\varepsilon^{-1}}\|_{\mathcal{H}^\alpha} = \mathcal{O}(\varepsilon^{2-8\kappa_0}) \tag{5.15}$$

$$\sup_{T\in[0,\tau^*]} |P_c v(T) - Z_\varepsilon(T) - e^{-T\varepsilon^{-2}} P_c v(0)| = \mathcal{O}(\varepsilon^{1-3\kappa_0}). \tag{5.16}$$

Proof. The mild solution of $v - P_c v$ is given by

$$(1-P_c)v(T) = e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon T}(1-P_c)v(0) + \int_0^T e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon (T-s)}(1-P_c)(\nu v(s) + \varepsilon^{-1}\vartheta v^2(s) - v^3(s))ds. \tag{5.17}$$

Define

$$v_+ := P_+ v \qquad v_- := P_- v \qquad v_c := P_c v.$$

Replacing v by $v_+ + v_- + v_c$ on the right-hand side of (5.17) gives us the following decomposition of $v - P_c v$:

$$\begin{aligned}
v_0 &:= \int_0^T e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon (T-s)}(2\varepsilon^{-1}\vartheta v_+ v_- + 6 v_c v_+ v_-)(s)\, ds \\
v_2 &:= \int_0^T e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon (T-s)}(\varepsilon^{-1}\vartheta {v_+}^2 + 3 v_c {v_+}^2)(s)\, ds \\
v_3 &:= \int_0^T e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon (T-s)}({v_+}^3)(s)\, ds \\
v_{-2} &:= \int_0^T e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon (T-s)}(\varepsilon^{-1}\vartheta {v_-}^2 + 3 v_c {v_-}^2)(s)\, ds \\
v_{-3} &:= \int_0^T e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon (T-s)}({v_-}^3)(s)\, ds \\
v_1 &:= e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon T} v_+(0) + \int_0^T e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon (T-s)}(2\vartheta\varepsilon^{-1} v_c v_+)(s)\, ds \\
&\quad + \int_0^T e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon (T-s)}(\nu v_+ + 3{v_+}^2 v_- + 3{v_c}^2 v_+)(s)\, ds \\
v_{-1} &:= e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon T} v_-(0) + \int_0^T e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon (T-s)}(2\vartheta\varepsilon^{-1} v_c v_-)(s)\, ds \\
&\quad + \int_0^T e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon (T-s)}(\nu v_- + 3 v_+ {v_-}^2 + 3{v_c}^2 v_-)(s)\, ds.
\end{aligned}$$

Using this decomposition we can write v^2 in (5.17) as

$$\begin{aligned}
v^2 &= (P_s v + P_s^c v)^2 \\
&= (\sum_{j=-3}^{3} P_s v_j)^2 + 2(P_s^c(v_+ + v_-) + v_c) \cdot \sum_{j=-3}^{3} P_s v_j + (P_s^c v)^2
\end{aligned}$$

and group the resulting terms together which results in the decomposition

$$
\begin{aligned}
\tilde{v}_0 &:= \int_0^T e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon(T-s)}\varepsilon^{-1}\vartheta\Big(\sum_{j=-3}^{3} P_s v_j P_s v_{-j} + v_c P_s v_0 \\
&\qquad\qquad + (P_s^c v)^2 + 2v_c P_s^c(v-v_c)\Big)(s)\, ds \\
\tilde{v}_2 &:= \int_0^T e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon(T-s)}\varepsilon^{-1}\vartheta\Big(\sum_{j=-1}^{3} P_s v_j P_s v_{2-j} \\
&\qquad\qquad + 2P_s^c v_+ P_s v_1 + 2P_s^c v_- P_s v_3 + 4v_c P_s v_2\Big)(s)\, ds \\
\tilde{v}_{-2} &:= \int_0^T e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon(T-s)}\varepsilon^{-1}\vartheta\Big(\sum_{j=-3}^{1} P_s v_j P_s v_{-2-j} \\
&\qquad\qquad + 2P_s^c v_+ P_s v_{-3} + 2P_s^c v_- P_s v_{-1} + 4v_c P_s v_{-2}\Big)(s)\, ds \\
\tilde{v}_1 &:= \Big(v_1 - e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon T} v_+(0) - \int_0^T e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon(T-s)}(2\vartheta\varepsilon^{-1} v_c v_+)(s)\, ds\Big) \\
&\quad + \int_0^T e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon(T-s)}\varepsilon^{-1}\vartheta\Big(\sum_{j=-2}^{3} P_s v_j P_s v_{1-j} + 2P_s^c v_+ P_s v_0 \\
&\qquad\qquad + 2P_s^c v_- P_s v_2 + 4v_c P_s v_1\Big)(s)\, ds \\
\tilde{v}_{-1} &:= \Big(v_{-1} - e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon T} v_-(0) \int_0^T e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon(T-s)}(2\vartheta\varepsilon^{-1} v_c v_-)(s)\, ds\Big) \\
&\quad + \int_0^T e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon(T-s)}\varepsilon^{-1}\vartheta\Big(\sum_{j=-3}^{2} P_s v_j P_s v_{-1-j} + 2P_s^c v_+ P_s v_{-2} \\
&\qquad\qquad + 2P_s^c v_- P_s v_0 + 4v_c P_s v_{-1}\Big)(s)\, ds \\
\tilde{v}_3 &:= v_3 + \int_0^T e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon(T-s)}\varepsilon^{-1}\vartheta\Big(\sum_{j=0}^{3} P_s v_j P_s v_{3-j} \\
&\qquad\qquad + 2P_s^c v_+ P_s v_2 + 4v_c P_s v_3\Big)(s)\, ds \\
\tilde{v}_{-3} &:= v_{-3} + \int_0^T e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon(T-s)}\varepsilon^{-1}\vartheta\Big(\sum_{j=-3}^{0} P_s v_j P_s v_{-3-j} \\
&\qquad\qquad + P_s^c v_- P_s v_{-2} + 4v_c P_s v_{-3}\Big)(s)\, ds,
\end{aligned}
$$

where we used that

$$v_c(v_+ + v_-) = v_c\Big(\sum_{j=-3}^{3} P_s v_j + P_s^c(v - v_c)\Big)$$

to decompose the $v_c v_+$ and $v_c v_-$ terms from v_1 and v_{-1}. In order to bound all those terms we essentially need only the following argument. Define for better readability

$$p_\varepsilon(k) := (1 - (\varepsilon k)^2)^2.$$

Using the simple inequality

$$\int_0^T e^{-\varepsilon^{-2}c(T-s)}ds \leq c^{-1}\varepsilon^2. \tag{5.18}$$

we get for all $f \in (0,T) \to \mathcal{H}^\alpha$, $j \in \mathbb{N}$:

$$\begin{aligned}
&\|P_s(\int_0^T e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon(T-s)}f\ ds)\ e^{-ij\varepsilon^{-1}x}\|^2_{\mathcal{H}^\alpha} \\
&= \|\int_0^T \int_S e^{-\varepsilon^{-2}p_\varepsilon(k)(T-s)}\mathcal{F}(f)(k)e^{ikx-j\varepsilon^{-1}}dkds\|^2_{\mathcal{H}^\alpha} \\
&= \int_{S-j\varepsilon^{-1}}(1+k^2)^\alpha \Big|\int_0^T e^{-\varepsilon^{-2}p_\varepsilon(k+j\varepsilon^{-1})(T-s)}\mathcal{F}(f)(k+j\varepsilon^{-1})ds\Big|^2 dk \\
&\leq \int_{S-j\varepsilon^{-1}}(1+k^2)^\alpha\ T_0 \int_0^T \Big|e^{-\varepsilon^{-2}p_\varepsilon(k+j\varepsilon^{-1})(T-s)}\mathcal{F}(f)(k+j\varepsilon^{-1})\Big|^2 ds\ dk \\
&= T_0 \int_0^T \int_{S-j\varepsilon^{-1}} e^{-2\varepsilon^{-2}p_\varepsilon(k+j\varepsilon^{-1})(T-s)}(1+k^2)^\alpha|\mathcal{F}(f)(k+j\varepsilon^{-1})|^2 dk\ ds \\
&\leq T_0 \int_0^T e^{-\varepsilon^{-2}c(T-s)} \int_{S-j\varepsilon^{-1}}(1+k^2)^\alpha|\mathcal{F}(f)(k+j\varepsilon^{-1})|^2 dk\ ds \\
&\leq T_0 \sup_{T\in[0,\tau^*]}(\int_{S-j\varepsilon^{-1}}(1+k^2)^\alpha|\mathcal{F}(f)(k+j\varepsilon^{-1})|^2 dk) \int_0^T e^{-\varepsilon^{-2}c(T-s)}\ ds \\
&\leq T_0 C\varepsilon^2 \sup_{T\in[0,\tau^*]}\|(P_s f)e^{-ij\varepsilon^{-1}}\|^2_{\mathcal{H}^\alpha},
\end{aligned} \tag{5.19}$$

where $S = (-\infty, -\frac{9}{8}\varepsilon^{-1}) \cup (-\frac{7}{8}\varepsilon^{-1}, \frac{7}{8}\varepsilon^{-1}) \cup (\frac{9}{8}\varepsilon^{-1}, \infty)$ and $c = \min_{k\in S}(p_\varepsilon(k)) > 0$. Therefore we get the bounds

$$\begin{aligned}
\|P_s v_0\|_{\mathcal{H}^\alpha} &\leq C(\varepsilon\vartheta + \varepsilon^2|v_c|)\|v_+ v_-\|_{\mathcal{H}^\alpha} \\
&\leq C(\varepsilon\vartheta + \varepsilon^2|v_c|)(\|v_+ e^{-i\varepsilon^{-1}x}\|^2_{\mathcal{H}^\alpha} + \|v_- e^{i\varepsilon^{-1}x}\|^2_{\mathcal{H}^\alpha}) \\
\|(P_s v_2)e^{-i2\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} &\leq C(\varepsilon\vartheta + \varepsilon^2|v_c|)\|(v_+ e^{-i\varepsilon^{-1}x})^2\|_{\mathcal{H}^\alpha} \\
&\leq C(\varepsilon\vartheta + \varepsilon^2|v_c|)\|v_+ e^{-i\varepsilon^{-1}x}\|^2_{\mathcal{H}^\alpha} \\
\|(P_s v_3)e^{-i3\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} &\leq C\varepsilon^2\|(v_+ e^{-i\varepsilon^{-1}x})^3\|_{\mathcal{H}^\alpha} \leq C\varepsilon^2\|v_+ e^{-i\varepsilon^{-1}x}\|^3_{\mathcal{H}^\alpha} \\
\|(P_s v_{-2})e^{i2\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} &\leq C(\varepsilon\vartheta + \varepsilon^2|v_c|)\|(v_- e^{i\varepsilon^{-1}x})^2\|_{\mathcal{H}^\alpha} \\
&\leq C(\varepsilon\vartheta + \varepsilon^2|v_c|)\|v_- e^{i\varepsilon^{-1}x}\|^2_{\mathcal{H}^\alpha} \\
\|(P_s v_{-3})e^{i3\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} &\leq C\varepsilon^2\|(v_- e^{i\varepsilon^{-1}x})^3\|_{\mathcal{H}^\alpha} \leq C\varepsilon^2\|v_- e^{i\varepsilon^{-1}x}\|^3_{\mathcal{H}^\alpha} \\
\|(P_s v_1)e^{-i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} &\leq \|P_s(e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon T}v_+(0))e^{-i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} + C\varepsilon\vartheta|v_c|\|v_+ e^{-i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} \\
&\quad + C\varepsilon^2(\nu\|v_+ e^{-i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} + 3\|{v_+}^2 v_- e^{-i(2-1)\varepsilon^{-1}x})\|_{\mathcal{H}^\alpha}) \\
&\leq \|P_s(e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon T}v_+(0))e^{-i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} + C\varepsilon\vartheta|v_c|\|v_+ e^{-i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} \\
&\quad + C\varepsilon^2(\nu\|v_+ e^{-i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} + 3\|v_+ e^{-i\varepsilon^{-1}x}\|^4_{\mathcal{H}^\alpha} + 3\|v_- e^{i\varepsilon^{-1}x}\|^2_{\mathcal{H}^\alpha})
\end{aligned}$$

$$\begin{aligned}
\|(P_s v_{-1})e^{i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} &\le \|P_s(e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon T}v_-(0))e^{i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} + C\varepsilon\vartheta|v_c|\|v_- e^{i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} \\
&\quad + C\varepsilon^2(\nu\|v_- e^{i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} + 3\|v_+ {v_-}^2 e^{i(2-1)\varepsilon^{-1}x})\|_{\mathcal{H}^\alpha}) \\
&\le \|P_s(e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon T}v_-(0))e^{-i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} + C\varepsilon\vartheta|v_c|\|v_- e^{i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} \\
&\quad + C\varepsilon^2(\nu\|v_- e^{i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} + 3\|v_- e^{i\varepsilon^{-1}x}\|^4_{\mathcal{H}^\alpha} + 3\|v_- e^{i\varepsilon^{-1}x}\|^2_{\mathcal{H}^\alpha})
\end{aligned}$$

and

$$\begin{aligned}
&\|(P_s(v_1 - e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon T}v_+(0)) - \int_0^T e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon(T-s)}(2\vartheta\varepsilon^{-1}v_c v_+)(s)\,ds)e^{-i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} \\
&\quad \le C\varepsilon^2(\nu\|v_+ e^{-i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} + 3\|v_+ e^{-i\varepsilon^{-1}x}\|^4_{\mathcal{H}^\alpha} + 3\|v_- e^{i\varepsilon^{-1}x}\|^2_{\mathcal{H}^\alpha}) \\
&\|(P_s(v_{-1} - e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon T}v_-(0))) - \int_0^T e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon(T-s)}(2\vartheta\varepsilon^{-1}v_c v_-)(s)\,ds e^{i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} \\
&\quad \le C\varepsilon^2(\nu\|v_- e^{i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} + 3\|v_- e^{i\varepsilon^{-1}x}\|^4_{\mathcal{H}^\alpha} + 3\|v_+ e^{-i\varepsilon^{-1}x}\|^2_{\mathcal{H}^\alpha}),
\end{aligned}$$

where we first used (5.19) and then Lemma 18 with the embedding (5.9). For any $f \in \mathcal{H}^\alpha$ we have

$$\begin{aligned}
&\|P_s(e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon T}f)e^{-ij\varepsilon^{-1}x}\|^2_{\mathcal{H}^\alpha} \\
&\quad = \int_{\mathbb{R}}(1+k^2)^\alpha\big|\mathcal{F}(P_s(e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon T}f)e^{-ij\varepsilon^{-1}x})(k)\big|^2 dk \\
&\quad = \int_{S-j\varepsilon^{-1}}(1+k^2)^\alpha\big|e^{-\varepsilon^{-2}p_\varepsilon(k+j\varepsilon^{-1})T}\mathcal{F}(f)(k+j\varepsilon^{-1})\big|^2 dk \\
&\quad \le e^{-\varepsilon^{-2}cT}\int_{\mathbb{R}}(1+k^2)^\alpha\big|\mathcal{F}(f)(k+j\varepsilon^{-1})\big|^2 dk = e^{-\varepsilon^{-2}cT}\|fe^{-ij\varepsilon^{-1}x}\|^2_{\mathcal{H}^\alpha},
\end{aligned} \tag{5.20}$$

so we can further bound $P_s v_1$ and $P_s v_{-1}$ by

$$\begin{aligned}
\|(P_s v_1(T))e^{-i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} &\le e^{-\varepsilon^{-2}cT}\|v_+(0)e^{-i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} + C\varepsilon|v_c|\|v_+ e^{-i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} \\
&\quad + C\varepsilon^2(\|v_+ e^{-i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} + \|v_+ e^{-i\varepsilon^{-1}x}\|^4_{\mathcal{H}^\alpha} + \|v_- e^{i\varepsilon^{-1}x}\|^2_{\mathcal{H}^\alpha})
\end{aligned}$$

and

$$\begin{aligned}
\|(P_s v_{-1}(T))e^{i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} &\le e^{-\varepsilon^{-2}cT}\|v_-(0)e^{i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} + C\varepsilon|v_c|\|v_- e^{i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} \\
&\quad + C\varepsilon^2(\|v_- e^{i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} + \|v_- e^{i\varepsilon^{-1}x}\|^4_{\mathcal{H}^\alpha} + \|v_+ e^{-i\varepsilon^{-1}x}\|^2_{\mathcal{H}^\alpha}).
\end{aligned}$$

Now (5.14) follows directly from the definition of τ^* in (5.12) and $e^{-\varepsilon^{-2}cT}$ being smaller than one. Even more we find

$$\begin{aligned}
&\sup_{T\in[0,\tau^*]} \|(P_s v_3(T))e^{-i3\varepsilon^{-1}}\|_{\mathcal{H}^\alpha} = \mathcal{O}(\varepsilon^{2-3\kappa_0}) \\
&\sup_{T\in[0,\tau^*]} \|(P_s v_{-3}(T))e^{i3\varepsilon^{-1}}\|_{\mathcal{H}^\alpha} = \mathcal{O}(\varepsilon^{2-3\kappa_0}) \\
&\sup_{T\in[0,\tau^*]} \|(P_s(v_1 - e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon T}v_+(0)) - \int_0^T e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon(T-s)}(2\vartheta\varepsilon^{-1}v_c v_+)(s)\,ds)e^{-i\varepsilon^{-1}}\|_{\mathcal{H}^\alpha} \\
&\qquad\qquad = \mathcal{O}(\varepsilon^{2-4\kappa_0}) \\
&\sup_{T\in[0,\tau^*]} \|(P_s(v_{-1} - e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon T}v_-(0)) - \int_0^T e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon(T-s)}(2\vartheta\varepsilon^{-1}v_c v_-)(s)\,ds)e^{i\varepsilon^{-1}}\|_{\mathcal{H}^\alpha} \\
&\qquad\qquad = \mathcal{O}(\varepsilon^{2-4\kappa_0}).
\end{aligned} \tag{5.21}$$

For $\tilde{v}_j$ we use the same calculation as in (5.19) to get

$$\|P_{s2}(\int_0^T e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon(T-s)}f\,ds)\,e^{-ij\varepsilon^{-1}x}\|^2_{\mathcal{H}^\alpha} \le CT_0\varepsilon^2 \sup_{T\in[0,\tau^*]}\|P_{s2}(f)e^{-ij\varepsilon^{-1}}\|^2_{\mathcal{H}^\alpha}.$$

From this follows that

$$\begin{aligned}
&\max_{|j|\le 3}\|P_{s2}\tilde{v}_j e^{-ij\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} \\
&\quad \le \|P_{s2}v_3 e^{-i3\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} + \|P_{s2}v_{-3}e^{i3\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} \\
&\qquad + \|(P_{s2}(v_1 - e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon T}v_+(0) - \int_0^T e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon(T-s)}(2\alpha\varepsilon^{-1}v_c v_+)(s)\,ds))e^{-i\varepsilon^{-1}}\|_{\mathcal{H}^\alpha} \\
&\qquad + \|(P_{s2}(v_{-1} - e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon T}v_-(0) - \int_0^T e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon(T-s)}(2\alpha\varepsilon^{-1}v_c v_-)(s)\,ds))e^{i\varepsilon^{-1}}\|_{\mathcal{H}^\alpha} \\
&\qquad + C\varepsilon\alpha(\max_{k+l=j}\|P_{s2}(P_s v_k P_s v_l)e^{-ij\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} + \max_{-3<j\le 3}\|P_{s2}(v_+ P_s v_{j-1})e^{-ij\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} \\
&\qquad + \max_{-3\le j<3}\|P_{s2}(v_- P_s v_{j+1})e^{-ij\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} + \max_{-3<j\le 3}\|P_{s2}(v_c P_s v_j)e^{-ij\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} \\
&\qquad + \|P_{s2}((P_s^c v)^2 + 2v_c P_s^c(v - v_c))\|_{\mathcal{H}^\alpha}).
\end{aligned} \tag{5.22}$$

Because

$$\begin{aligned}
\|(P_{s2}f)e^{iyx}\|^2_{\mathcal{H}^\alpha} &= \int_{\mathbb{R}} (1+k^2)^\alpha |\mathcal{F}((P_{s2}f)e^{iyx})(k)|^2 dk \\
&= \int_{S2+y} (1+k^2)^\alpha |\mathcal{F}(f)(k-y)|^2 dk \\
&\le \int_{S+y} (1+k^2)^\alpha |\mathcal{F}(f)(k-y)|^2 dk = \|(P_s f)e^{iyx}\|^2_{\mathcal{H}^\alpha}
\end{aligned}$$

for any $f \in \mathcal{H}^\alpha$, $y \in \mathbb{R}$, the first four terms are bounded by (5.21) and together with (5.20) and Lemma 19 we get

$$\begin{aligned}
\max_{k+l=j} \|P_{s2}(v_k v_l)e^{-ij\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} &\le C \max_{k+l=j} (\|(P_s v_k)e^{-ik\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} \|(P_s v_l)e^{-il\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha}) \\
\max_{-3\le j<3} \|P_{s2}(v_- v_{j+1})e^{-ij\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} &\le C \max_{-3\le j<3} (\|v_- e^{i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} \|(P_s v_{j+1})e^{-i(j+1)\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha}) \\
\max_{-3\le j<3} \|P_{s2}(v_+ v_{j-1})e^{-ij\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} &\le C \max_{-3< j\le 3} (\|v_+ e^{-i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} \|(P_s v_{j-1})e^{-i(j-1)\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha}) \\
\max_{-3< j\le 3} \|P_{s2}(v_c P_s v_j)e^{-ij\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} &\le C \max_{-3< j\le 3} (|v_c| \; \|(P_s v_j)e^{-ij\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha}),
\end{aligned}$$

which leaves the last term. It is actually equal to zero as we see from the following:

The Fourier transform of the product of two functions $f, g : \mathbb{R} \to \mathbb{C}$ whose Fourier transforms fulfil $\mathrm{supp}(\mathcal{F}(f)) \subset [a,b]$ and $\mathrm{supp}(\mathcal{F}g) \subset [c,d]$ is given by

$$\begin{aligned}
\mathcal{F}(fg)(k) = (\mathcal{F}(f) * \mathcal{F}(g))(k) &= \int_{\mathbb{R}} \mathcal{F}(f)(k-l)\mathcal{F}(g)(l)\; dl \\
&= \int_{\mathbb{R}} \mathcal{X}_{[a,b]}(k-l)\mathcal{X}_{[c,d]}(l)\mathcal{F}(f)(k-l)\mathcal{F}(g)(l)\; dl,
\end{aligned}$$

from which it follows that

$$\mathrm{supp}(\mathcal{F}(fg)) \subset [a+c, b+d]. \tag{5.23}$$

The last term of (5.22) can be written as

$$P_{s2}((P_s^c v)^2 + 2v_c P_s^c(v - v_c)) = P_{s2}((P_s^c(v-v_c))^2 + 4v_c P_s^c(v-v_c) + v_c^2).$$

We have $\mathrm{supp}\{\mathcal{F}(P_s^c(v-v_c))\} \subset [-\frac{9}{8}, -\frac{7}{8}] \cup [\frac{7}{8}, \frac{9}{8}]$ and from (5.23) we get

$$\mathrm{supp}\{\mathcal{F}(P_s^c(v-v_c))^2\} \subset [-\tfrac{5}{4}, -\tfrac{3}{4}] \cup [-\tfrac{1}{4}, \tfrac{1}{4}] \cup [\tfrac{3}{4}, \tfrac{5}{4}].$$

The intersection of $S2$ with these supports being empty yields

$$P_{s2}(P_s^c(v-v_c))^2 = 0,$$

which concludes the proof of (5.15).

The projection $P_c v$ of the mild solution v is given by

$$\begin{aligned}
P_c v(T) &= e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon T} P_c v(0) + \int_0^T e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon (T-s)} \varepsilon^{-1}\sigma d\tilde\beta \\
&\quad + \int_0^T e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon (T-s)} (\nu P_c v(s) + \varepsilon^{-1}\vartheta P_c v^2(s) - P_c v^3(s)) ds. \\
&= e^{-\varepsilon^{-2}T} P_c v(0) + \int_0^T e^{-\varepsilon^{-2}(T-s)} \varepsilon^{-1}\sigma d\tilde\beta \\
&\quad + \int_0^T e^{-\varepsilon^{-2}(T-s)} (\nu P_c v(s) + \varepsilon^{-1}\vartheta (P_c v)^2(s) - (P_c v)^3(s)) ds.
\end{aligned}$$

Where we used that $\lim_{x\to\infty} v(x)^m = (\lim_{x\to\infty} v(x))^m$. Thus by using the inequality (5.18) again, we obtain (5.16):

$$\sup_{T\in[0,\tau^*]} |P_c v(T) - Z_\varepsilon(T) - e^{-T\varepsilon^{-2}} P_c v(0)|$$
$$\leq CT_0\varepsilon^2(2 + |\nu| + \varepsilon^{-1}|\vartheta|) \sup_{T\in[0,\tau^*]} (\sum_{m=1}^{3} |P_c v(T)|^m) = \mathcal{O}(\varepsilon^{1-3\kappa_0})$$

□

Being prepared with these estimates we can approximate the stochastic differentials of the projections P_a, P_Φ and P_Ψ onto the peaks of the Fourier transform of $(1-P_c)v$. As well as of the remaining term when we subtract Z_ε from $P_c v$.

5.2.4 Approximation of main components

For simplicity of presentation let us represent the terms of interest by the following functions:

$$\begin{aligned} a(T) &:= P_a v(T) e^{-i\varepsilon^{-1}x} \\ \Phi(T) &:= \varepsilon^{-1} P_\Phi(v(T) - e^{-T\varepsilon^{-2}\mathcal{L}_\varepsilon} v(0)) e^{-i2\varepsilon^{-1}x} \\ \Psi(T) &:= \varepsilon^{-1} P_\Psi(v(T) - e^{-T\varepsilon^{-2}\mathcal{L}_\varepsilon} v(0)) \\ \Psi_c(T) &:= \varepsilon^{-1} P_c(v(T) - Z_\varepsilon(T) - e^{-T\varepsilon^{-2}\mathcal{L}_\varepsilon} v(0)). \end{aligned} \tag{5.24}$$

The projections are shifted such that their centre is at zero and rescaled such that their $\mathcal{H}^\alpha$-norm is approximately of order one.

Lemma 24. *The derivatives of a, Φ, Ψ and Ψ_c are given by*

$$\begin{aligned} da = (&-\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1} a + \nu a + 2\vartheta P_\Psi(\bar{a}\Phi + a\Psi + a\Psi_c))dT \\ &- (P_\Psi(3a|a|^2 - 3aZ_\varepsilon^2 + \varepsilon^{-1}2\vartheta a Z_\varepsilon) + R_1)dT \end{aligned} \tag{5.25}$$

$$d\Phi = (-\mathcal{L}_\varepsilon^{+2}\varepsilon^{-2}\Phi + \varepsilon^{-2}\vartheta P_\Psi a^2 + R_2)dT \tag{5.26}$$

$$d\Psi = (-\mathcal{L}_\varepsilon \varepsilon^{-2}\Psi + \varepsilon^{-2}\vartheta P_\Psi |a|^2 + R_3)dT \tag{5.27}$$

$$d\Psi_c = (-\varepsilon^{-2}\Psi_c + \varepsilon^{-2}\vartheta Z_\varepsilon^2 + R_4)dT \tag{5.28}$$

where R_1, R_2, R_3 and R_4 are stochastic processes with

$$\int_0^{\tau^*} \|R_1\|_{\mathcal{H}^\alpha} ds = \mathcal{O}(\varepsilon^{1-9\kappa_0}),$$
$$\int_0^{\tau^*} \|R_2\|_{\mathcal{H}^\alpha} + \|R_3\|_{\mathcal{H}^\alpha} + \|R_4\|_\infty ds = \mathcal{O}(\varepsilon^{-1-9\kappa_0}).$$

Proof. We split v into
$$v = v_A + v_B,$$
where
$$\begin{aligned}
v_A &= ae^{i\varepsilon^{-1}x} + \bar{a}e^{-i\varepsilon^{-1}x} + Z_\varepsilon \\
v_B &= \varepsilon\Phi e^{i2\varepsilon^{-1}x} + \varepsilon\bar{\Phi}e^{-i2\varepsilon^{-1}x} + \varepsilon\Psi + \varepsilon\Psi_c + P_{s2}(v - e^{-\varepsilon^{-2}T\mathcal{L}_\varepsilon}v(0)) + e^{-T\varepsilon^{-2}\mathcal{L}_\varepsilon}P_s v(0).
\end{aligned}$$

Note that $v_A = P_s^c v$ and $v_B = P_s v$. Now we multiply v with itself to get
$$\begin{aligned}
v^2 &= 2(ae^{i\varepsilon^{-1}x} + \bar{a}e^{-i\varepsilon^{-1}x} + Z_\varepsilon)(\varepsilon\Phi e^{i2\varepsilon^{-1}x} + \varepsilon\bar{\Phi}e^{-i2\varepsilon^{-1}x} + \varepsilon\Psi + \varepsilon\Psi_c) \\
&\quad + (ae^{i\varepsilon^{-1}x} + \bar{a}e^{-i\varepsilon^{-1}x} + Z_\varepsilon)^2 + r_1 \qquad (5.29)\\
v^3 &= (ae^{i\varepsilon^{-1}x} + \bar{a}e^{-i\varepsilon^{-1}x} + Z_\varepsilon)^3 + r_2
\end{aligned}$$

with
$$\begin{aligned}
r_1 &= v_B^2 + v_A(\ P_{s2}(v - e^{-\varepsilon^{-2}T\mathcal{L}_\varepsilon}v(0)) + e^{-T\varepsilon^{-2}\mathcal{L}_\varepsilon}P_s v(0)\ ) \\
r_2 &= v_B^3 + 3v_A v_B^2 + 3v_A^2 v_B.
\end{aligned}$$

For $m \in \{\text{'}a\text{'}, \text{'}\Phi\text{'}, \text{'}\Psi\text{'}\}$ and
$$n_a := 1 \qquad\qquad n_\Phi := 2 \qquad\qquad n_\Psi := 0$$

we have
$$\begin{aligned}
&\|P_m(v_A^l v_B^k)e^{-in_m\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} \\
&= \|P_m\Big((ae^{i\varepsilon^{-1}x} + \bar{a}e^{-i\varepsilon^{-1}x} + Z_\varepsilon)^l(\varepsilon\Psi_c + \sum_{j=-3}^{3} P_s v_j(T))^k\Big)e^{-in_m\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} \\
&\leq \sum_{\substack{l_1+l_2+l_3=l \\ \sum_{j=-3}^4 k_j=k}} \|P_m\Big((ae^{i\varepsilon^{-1}x})^{l_1}(\bar{a}e^{-i\varepsilon^{-1}x})^{l_2}(Z_\varepsilon)^{l_3} \\
&\qquad\qquad \times \prod_{j=-3}^{3}(P_s v_j)^{k_j}(\varepsilon\Psi_c)^{k_4}\Big)e^{-in_m\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} \\
&= \sum_{\substack{l_1+l_2+l_3=l \\ \sum_{j=-3}^4 k_j=k}} |Z_\varepsilon|^{l_3}|\varepsilon\Psi_c|^{k_4}\|P_m\Big((a)^{l_1}(\bar{a})^{l_2} \\
&\qquad\qquad \times \prod_{j=-3}^{3}((P_s v_j)e^{-ij\varepsilon^{-1}})^{k_j}e^{i(l_1-l_2+\sum_{j=-3}^3 jk_j)\varepsilon^{-1}x}\Big)e^{-in_m\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha},
\end{aligned}$$

where v_j is the decomposition from Lemma 23. Every summand has the form
$$\|P_m(fe^{ij\varepsilon^{-1}x})e^{-in_m\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha}$$

with $f \in \mathcal{H}^\alpha$ and $j \in \mathbb{N}$. Terms of this kind can be bounded by

$$\begin{aligned}
&\|P_m(fe^{ij\varepsilon^{-1}x})e^{-in_m\varepsilon^{-1}x}\|^2_{\mathcal{H}^\alpha} \\
&\quad= \int_{\mathbb{R}} (1+k^2)^\alpha |\mathcal{F}(P_m(fe^{ij\varepsilon^{-1}x})e^{-in_m\varepsilon^{-1}x})(k)|^2 dk \\
&\quad= \int_{\mathbb{R}} (1+k^2)^\alpha |\mathcal{F}(P_m(fe^{ij\varepsilon^{-1}x}))(k+n_m\varepsilon^{-1})|^2 dk \\
&\quad= \int_{-1/4\varepsilon^{-1}}^{1/4\varepsilon^{-1}} (1+k^2)^\alpha |\mathcal{F}(f)(k+(n_m-j)\varepsilon^{-1})|^2 dk \\
&\quad= \int_{((n_m-j)-1/4)\varepsilon^{-1}}^{((n_m-j)+1/4)\varepsilon^{-1}} (1+(k-(n_m-j)\varepsilon^{-1})^2)^\alpha |\mathcal{F}(f)(k)|^2 dk \\
&\quad\leq \int_{((n_m-j)-1/4)\varepsilon^{-1}}^{((n_m-j)+1/4)\varepsilon^{-1}} (1+k^2)^\alpha |\mathcal{F}(f)(k)|^2 dk = \|f\|^2_{\mathcal{H}^\alpha},
\end{aligned} \tag{5.30}$$

since

$$\forall j \in \mathbb{N}: \quad \max_{|k|\leq 1/4} |k| \leq \min_{|k|\leq 1/4} |j+k|.$$

Therefore we arrive at

$$\begin{aligned}
&\|P_m(v_A^l v_B^k)e^{-in_m\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} \\
&\quad\leq \sum_{\substack{l_1+l_2+l_3=l \\ \sum_{j=-3}^4 k_j = k}} |Z_\varepsilon|^{l_3} |\varepsilon\Psi_c|^{k_4} \|(a)^{l_1}(\bar{a})^{l_2} \prod_{j=-3}^{3} ((P_s v_j)e^{-ij\varepsilon^{-1}})^{k_j}\|_{\mathcal{H}^\alpha} \\
&\quad\leq \sum_{\substack{l_1+l_2+l_3=l \\ \sum_{j=-3}^4 k_j = k}} \Big(\|a\|^{l_1}_{\mathcal{H}^\alpha} \, \|\bar{a}\|^{l_2}_{\mathcal{H}^\alpha} \, |Z_\varepsilon|^{l_3} \prod_{j=-3}^{3} \|((P_s v_j)e^{-ij\varepsilon^{-1}})^{k_j}\|_{\mathcal{H}^\alpha} |\varepsilon\Psi_c|^{k_4} \Big).
\end{aligned}$$

Repeating the same steps for $v_A P_{s2}(v - e^{-\varepsilon^{-2}T\mathcal{L}_\varepsilon} v(0))$ and $v_A e^{-T\varepsilon^{-2}\mathcal{L}_\varepsilon} P_s v(0)$ leads to

$$\|P_m(v_A P_{s2} v)e^{-in_m\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} \leq (\|a\|_{\mathcal{H}^\alpha} + \|\bar{a}\|_{\mathcal{H}^\alpha} + |Z_\varepsilon|) \sum_{j=-3}^{3} \|((P_{s2}\tilde{v}_j)e^{-ij\varepsilon^{-1}})^{k_j}\|_{\mathcal{H}^\alpha}$$

and

$$\begin{aligned}
&\|P_m(v_A e^{-T\varepsilon^{-2}\mathcal{L}_\varepsilon}(P_s - P_{s2})v(0))e^{-in_m\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} \\
&\quad\leq (\|a\|_{\mathcal{H}^\alpha} + \|\bar{a}\|_{\mathcal{H}^\alpha} + |Z_\varepsilon|)(\|(e^{-T\varepsilon^{-2}\mathcal{L}_\varepsilon} P_s v_+(0))e^{-i\varepsilon^{-1}}\|_{\mathcal{H}^\alpha} \\
&\qquad\qquad + \|(e^{-T\varepsilon^{-2}\mathcal{L}_\varepsilon} P_s v_-(0))e^{i\varepsilon^{-1}}\|_{\mathcal{H}^\alpha}).
\end{aligned} \tag{5.31}$$

We recall from Lemma 23 that (with $\alpha \leq 1$)

$$\sup_{T\in[0,\tau^*]} \|(P_s v_j(T))e^{-ij\varepsilon^{-1}}\|_{\mathcal{H}^\alpha} = \mathcal{O}(\varepsilon^{1-4\kappa_0})$$
$$\sup_{T\in[0,\tau^*]} \|(P_{s2}\tilde{v}_j(T))e^{-ij\varepsilon^{-1}}\|_{\mathcal{H}^\alpha} = \mathcal{O}(\varepsilon^{2-8\kappa_0})$$
$$\sup_{T\in[0,\tau^*]} |\varepsilon\Psi_c| = \sup_{T\in[0,\tau^*]} |P_c(v(T) - e^{-T\varepsilon^{-2}\mathcal{L}_\varepsilon}v(0)) - Z_\varepsilon(T)| = \mathcal{O}(\varepsilon^{1-2\kappa_0}).$$

Also

$$\|\bar{a}\|_{\mathcal{H}^\alpha} = \|a\|_{\mathcal{H}^\alpha} = \|P_a v e^{-i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} \leq \|P_+ v e^{-i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} \tag{5.32}$$

and the OU-process can be bounded by

$$\sup_{T\in[0,\tau^*]} |Z_\varepsilon(T)| = \mathcal{O}(\varepsilon^{-\gamma}) \tag{5.33}$$

for all positive $\gamma \in \mathbb{R}$. For a proof of this well-known result see [BM13] p. 9 (Lemma 14). Therefore

$$\sup_{T\in[0,\tau^*]} \|P_m(v_A^l v_B^k)e^{-in_m\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} = \mathcal{O}(\varepsilon^{k-\kappa_0(l+4k)})$$

and

$$\sup_{T\in[0,\tau^*]} \|P_m(v_A P_{s2} v)e^{-in_m\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} = \mathcal{O}(\varepsilon^{2-9\kappa_0}).$$

Following the same steps as in (5.20) we derive

$$\|(e^{-\varepsilon^{-2}T\mathcal{L}_\varepsilon} P_s v_+(0))e^{-i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} \leq e^{\varepsilon^{-2}cT}\|(v_+(0)e^{-i\varepsilon^{-1}x})\|_{\mathcal{H}^\alpha}$$
$$\|(e^{-\varepsilon^{-2}T\mathcal{L}_\varepsilon} P_s v_-(0))e^{i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} \leq e^{\varepsilon^{-2}cT}\|(v_-(0)e^{i\varepsilon^{-1}x})\|_{\mathcal{H}^\alpha}\ .$$

So it follows from (5.31) that

$$\int_0^{\tau^*} \|P_m(v_A e^{-t\varepsilon^{-2}\mathcal{L}_\varepsilon} P_s v(0))e^{-in_m\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} dt = \mathcal{O}(\varepsilon^{2-2\kappa_0}).$$

Therefore we can bound the integral in time of r_1 and r_2 by

$$\int_0^{\tau^*} \|(P_m r_1)e^{-in_m\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} dt = \mathcal{O}(\varepsilon^{2-9\kappa_0}) \tag{5.34}$$
$$\int_0^{\tau^*} \|(P_m r_2)e^{-in_m\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} dt = \mathcal{O}(\varepsilon^{1-6\kappa_0}). \tag{5.35}$$

Inserting (5.29) into the mild solution of v yields

$$\begin{aligned}
d(ae^{i\varepsilon^{-1}x}) &= -\varepsilon^{-2}\mathcal{L}_\varepsilon P_a v + \nu P_a v + \varepsilon^{-1}\alpha P_a(v^2) - P_a(v^3)dT \\
&= (-\varepsilon^{-2}\mathcal{L}_\varepsilon a e^{i\varepsilon^{-1}x} + \nu a e^{i\varepsilon^{-1}x} + 2\vartheta P_a((\bar{a}\Phi + a\Psi + a\Psi_c + \varepsilon^{-1}aZ_\varepsilon)e^{i\varepsilon^{-1}x}) \\
&\quad - 3P_a((a|a|^2 + aZ_\varepsilon^2)e^{i\varepsilon^{-1}x}) + P_a(\tilde{R}_1 e^{i\varepsilon^{-1}x} + \tilde{R}_2 e^{i2\varepsilon^{-1}x} + \tilde{R}_3)\ dT \\
d(\Phi e^{i2\varepsilon^{-1}x}) &= -\varepsilon^{-2}\mathcal{L}_\varepsilon P_\Phi v + \varepsilon^{-2}\vartheta P_\Phi(a^2 e^{i2\varepsilon^{-1}x}) + \varepsilon^{-1}P_\Phi(\tilde{R}_1 e^{i\varepsilon^{-1}x} + \tilde{R}_2 e^{i2\varepsilon^{-1}x} + \tilde{R}_3)\ dT \\
d\Psi &= -\varepsilon^{-2}\mathcal{L}_\varepsilon P_\Psi v + \varepsilon^{-2}\vartheta P_\Psi(|a|^2) + \varepsilon^{-1}P_\Psi(\tilde{R}_1 e^{i\varepsilon^{-1}x} + \tilde{R}_2 e^{i2\varepsilon^{-1}x} + \tilde{R}_3)\ dT \\
d\Psi_c &= -\varepsilon^{-2}\Psi_c + \varepsilon^{-2}\vartheta Z_\varepsilon^2 + R_4\ dT,
\end{aligned}$$

where

$$\begin{aligned}
\tilde{R}_1 &= (\varepsilon^{-1}\vartheta P_a r_1 - P_a r_2)e^{-i\varepsilon^{-1}x} \\
\tilde{R}_2 &= (\varepsilon^{-2}\vartheta P_\Phi r_1 - \varepsilon^{-1}P_\Phi r_2)e^{-i2\varepsilon^{-1}x} \\
&\quad + \nu\Phi + 2\varepsilon^{-1}\vartheta Z_\varepsilon \Phi - 3a^2 Z_\varepsilon \\
\tilde{R}_3 &= \varepsilon^{-2}\vartheta P_\Psi r_1 - \varepsilon^{-1}P_\Psi r_2 \\
&\quad + \nu\Psi + \varepsilon^{-1}\vartheta\Psi Z_\varepsilon + 6|a|^2 Z_\varepsilon \\
R_4 &= \varepsilon^{-2}\vartheta P_c r_1 - \varepsilon^{-1}P_c r_2 + \nu\Psi_c - \varepsilon^{-1}Z_\varepsilon^3.
\end{aligned}$$

We omitted the following terms which, using property (5.23) that confines the support of the Fourier transform for products of functions, are all equal to zero:

$$\begin{aligned}
P_a(a^3e^{i3\varepsilon^{-1}x} + a^2e^{i2\varepsilon^{-1}x} + |a|^2) &= 0 \\
P_\Phi((\bar{a}\Phi + a\Psi + a\Psi_c + \varepsilon^{-1}aZ_\varepsilon + a|a|^2 + aZ_\varepsilon^2)e^{i\varepsilon^{-1}x} + a^3e^{i3\varepsilon^{-1}x} + |a|^2) &= 0 \\
P_\Psi((\bar{a}\Phi + a\Psi + a\Psi_c + \varepsilon^{-1}aZ_\varepsilon + a|a|^2 + aZ_\varepsilon^2)e^{i\varepsilon^{-1}x} + a^3e^{i3\varepsilon^{-1}x} + a^2e^{i2\varepsilon^{-1}x})) &= 0
\end{aligned}$$

Also from

$$P_iP_j = \begin{cases} 0 & \text{if } i \neq j \\ P_i & \text{if } i = j \end{cases} \tag{5.36}$$

for $i, j \in \{\text{'}a\text{'}, \text{'}\Phi\text{'}, \text{'}\Psi\text{'}\}$, in addition with (5.23), it follows that

$$\begin{aligned}
P_a(\tilde{R}_2e^{i2\varepsilon^{-1}x} + \tilde{R}_3) &= 0 \\
P_\Phi(\tilde{R}_1e^{i\varepsilon^{-1}x} + \tilde{R}_3) &= 0 \\
P_a(\tilde{R}_1e^{i\varepsilon^{-1}x} + \tilde{R}_2e^{i2\varepsilon^{-1}x}) &= 0.
\end{aligned}$$

Now we define (for $j \in \{1, 2, 3\}$)

$$R_j := P_\Psi(\tilde{R}_j).$$

Because of

$$P_a(fe^{i\varepsilon^{-1}x}) = \int_{\frac{3}{4}\varepsilon^{-1}}^{\frac{5}{4}\varepsilon^{-1}} \mathcal{F}(f)(k - \varepsilon^{-1})e^{ikx}dk = (P_\Psi f)e^{i\varepsilon^{-1}x} \tag{5.37}$$

and

$$P_\Phi(fe^{i2\varepsilon^{-1}x}) = \int_{\frac{7}{4}\varepsilon^{-1}}^{\frac{9}{4}\varepsilon^{-1}} \mathcal{F}(f)(k - 2\varepsilon^{-1})e^{ikx}dk = (P_\Psi f)e^{i2\varepsilon^{-1}x} \tag{5.38}$$

the only thing left is to prove the bounds on R_1 to R_4. From (5.38) and (5.37) we get

$$\begin{aligned} R_1 &= P_a(\tilde{R}_1 e^{i\varepsilon^{-1}x})e^{-i\varepsilon^{-1}x} = \varepsilon^{-1}\vartheta(P_a r_1)e^{-i\varepsilon^{-1}x} + (P_a r_2)e^{-i\varepsilon^{-1}x} \\ R_2 &= P_\Phi(\tilde{R}_2 e^{i2\varepsilon^{-1}x})e^{-i2\varepsilon^{-1}x} \\ &= \varepsilon^{-1}\vartheta(P_\Phi r_1)e^{-i2\varepsilon^{-1}x} + (P_\Phi r_2)e^{-i2\varepsilon^{-1}x} + \nu\Phi + 2\varepsilon^{-1}\vartheta Z_\varepsilon \Phi - 3P_\Psi(a^2 Z_\varepsilon) \\ R_3 &= P_\Psi(\tilde{R}_2) = \varepsilon^{-1}\vartheta P_\Psi r_1 + P_\Psi r_2 + \nu\Psi + \varepsilon^{-1}\vartheta\Psi Z_\varepsilon + 6P_\Psi(|a|^2 Z_\varepsilon). \end{aligned}$$

So we have

$$\begin{aligned} \int_0^T \|R_1\|_{\mathcal{H}^\alpha}dt &\leq \varepsilon^{-1}\vartheta\int_0^T \|(P_a r_1)e^{-i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha}dt + \int_0^T \|(P_a r_2)e^{-i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha}dt \\ \int_0^T \|R_2\|_{\mathcal{H}^\alpha}dt &\leq \varepsilon^{-1}\vartheta\int_0^T \|(P_\Phi r_1)e^{-i2\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha}dt + \int_0^T \|(P_\Phi r_2)e^{-i2\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha}dt \\ &\quad + CT\varepsilon^{-1}\sup_{s\in[0,T]}\|\Phi(s) + Z_\varepsilon(s)\Phi(s) + P_\Psi(a(s)^2 Z_\varepsilon(s))\|_{\mathcal{H}^\alpha} \\ \int_0^T \|R_3\|_{\mathcal{H}^\alpha}dt &\leq \varepsilon^{-1}\vartheta\int_0^T \|P_\Psi r_1\|_{\mathcal{H}^\alpha}dt + \int_0^T \|P_\Psi r_2\|_{\mathcal{H}^\alpha}dt \\ &\quad + CT\varepsilon^{-1}\sup_{s\in[0,T]}\|\Psi(s) + Z_\varepsilon(s)\Psi(s) + P_\Psi(|a(s)|^2 Z_\varepsilon(s))\|_{\mathcal{H}^\alpha}. \end{aligned}$$

The r_1 and r_2 terms are taken care of by (5.34), (5.35). The other terms are bounded by

$$\|\Phi\|_{\mathcal{H}^\alpha} = \|P_\Phi((\sum_{j=-3}^{3}(P_s v_j)e^{-ij\varepsilon^{-1}x})e^{ij\varepsilon^{-1}x})e^{-i2\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} \leq \sum_{j=-3}^{3}\|P_s v_j e^{-ij\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} \tag{5.39}$$

$$\|\Psi\|_{\mathcal{H}^\alpha} = \|P_\Psi((\sum_{j=-3}^{3}(P_s v_j)e^{-ij\varepsilon^{-1}x})e^{ij\varepsilon^{-1}x})\|_{\mathcal{H}^\alpha} \leq \sum_{j=-3}^{3}\|P_s v_j e^{-ij\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} \tag{5.40}$$

$$\|P_\Psi a^2\|_{\mathcal{H}^\alpha} + \|P_\Psi |a|^2\|_{\mathcal{H}^\alpha} \leq \|a^2\|_{\mathcal{H}^\alpha} + \|a\bar{a}\|_{\mathcal{H}^\alpha} \leq C\|v_+ e^{-i\varepsilon^{-1}x}\|^2_{\mathcal{H}^\alpha}$$

and (5.33). Finally R_4 is bounded by

$$\begin{aligned} |R_4| &= \varepsilon^{-2}\vartheta|P_c r_1| + \varepsilon^{-1}|P_c r_2| + \nu|\Psi_c| + \varepsilon^{-1}|Z_\varepsilon|^3 \\ &= \varepsilon^{-2}\vartheta|\varepsilon\Psi_c|^2 + \varepsilon^{-1}|(\varepsilon\Psi_c)^3 + (\varepsilon\Psi_c)^2 Z_\varepsilon + \varepsilon\Psi_c Z_\varepsilon^2| + \nu|\Psi_c| + \varepsilon^{-1}|Z_\varepsilon|^3 \end{aligned}$$

Therefore the bounds on R_1, R_2, R_3 and R_4 are valid. □

Now we see why the bound in (5.15) of the projection onto $S2$ is needed. Calculation (5.30) could not be done without the gaps between the different projections P_a, P_Ψ and P_Φ, which are handled by equation (5.15).
Calculation (5.30) takes care of the "tails" of other peaks v_j or $\tilde{v}_j$ that reach within

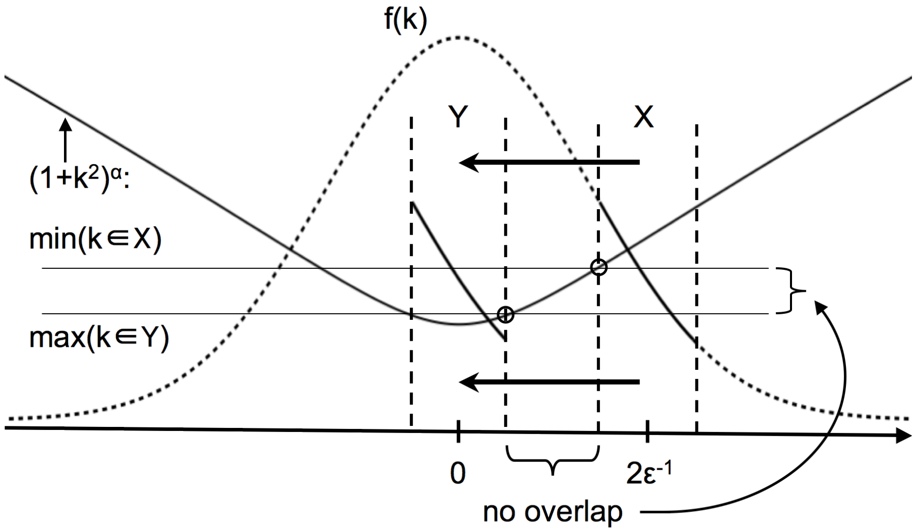

Figure 5.4 Estimating the "tails" of v_j or $\tilde{v}_j$

the range of a specific projection. The idea is illustrated in Figure 5.4 which shows a single peak $f(k)$ centred around zero that is projected by P_Φ onto the interval $[7/4\varepsilon^{-1}, 9/4\varepsilon^{-1}]$ and shifted such that the support of the projection is again centred around origin. Since the original support $X := [7/4\varepsilon^{-1}, 9/4\varepsilon^{-1}]$ and the shifted support $Y := [-1/4\varepsilon^{-1}, 1/4\varepsilon^{-1}]$ do not overlap, the maximum of $k^2 \in Y$ is smaller than the minimum of $k^2 \in X$ and thus the $\mathcal{H}^\alpha$-norm of the shifted projection is bounded by $\|P_\Phi f\|_{\mathcal{H}^\alpha} \leq \|f\|_{\mathcal{H}^\alpha}$.
Next we exchange the coupling terms in the SDE (5.25) for the dominant part $a(t)$ of the spectrum.

5.2.5 Exchange of the coupling terms

If we integrate the SDEs for Φ and Ψ which we obtained from the last Lemma, then we see that $\mathcal{L}_\varepsilon^{+2}\Phi$ is approximately $\vartheta P_\Psi a^2$ (up until order ε) and similarly for $\mathcal{L}_\varepsilon \Psi$. In order to replace $a\Phi$ (or $a\Psi$) we need two steps. Relate $\overline{a}\Phi$ to $\overline{a}\mathcal{L}_\varepsilon^{+2}\Phi$ and then by Itô formula $\overline{a}\mathcal{L}_\varepsilon^{+2}\Phi$ to $\overline{a}a^2$. The same approach also works with the other coupling terms $\varepsilon^{-1}aZ_\varepsilon$ and $a\Psi_c$, but without the need for taking care of the operator since it takes the form of a real number. The result is the Lemma below.

Lemma 25. *With a, Φ, Ψ, Ψ_c and Z_ε as defined in* (5.24) *and* (5.13), *there exist* *stochastic processes R_i, $i \in \{5,6,7,8,9,10\}$ such that*

$$\int_0^T e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}(9\bar{a}\Phi - \vartheta|a|^2 a)ds = \int_0^T R_5\, ds + R_6 \tag{5.41}$$

$$\int_0^T e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}(\bar{a}\Psi - \vartheta|a|^2 a)ds = \int_0^T R_7\, ds + R_8 \tag{5.42}$$

$$\begin{aligned}\| \int_0^T & e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}(\varepsilon^{-1} a Z_\varepsilon - a Z_\varepsilon^2)ds \\ & - \int_0^T e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}\sigma a d\beta \big\|_{\mathcal{H}^\alpha} = R_9\end{aligned} \tag{5.43}$$

$$\| \int_0^T e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}(a\Psi_c - \vartheta a Z_\varepsilon^2)ds\|_{\mathcal{H}^\alpha} = R_{10} \tag{5.44}$$

and the following bounds hold:

$$\int_0^{\tau^*} \|R_5\|_{\mathcal{H}^\alpha} + \|R_7\|_{\mathcal{H}^\alpha} ds = \mathcal{O}(\varepsilon^{\alpha - 10\kappa_0})$$

$$\sup_{T\in[0,\tau^*]} (\|R_6\|_{\mathcal{H}^\alpha} + \|R_8\|_{\mathcal{H}^\alpha} + \|R_9\|_{\mathcal{H}^\alpha} + \|R_{10}\|_{\mathcal{H}^\alpha}) = \mathcal{O}(\varepsilon^{1-10\kappa_0})$$

Proof. We split the left-hand sides of (5.41) and (5.42) into two parts each in order to bound them separately:

$$\begin{aligned}
&e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}(9\bar{a}\Phi - \vartheta|a|^2 a) \\
&\quad = e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}(9\bar{a}\Phi - \bar{a}\mathcal{L}_\varepsilon^{+2}\Phi) \\
&\qquad + e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}(\bar{a}\mathcal{L}_\varepsilon^{+2}\Phi - \vartheta a|a|^2) =: \theta_1 + \theta_2 \\
&e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}(a\Psi - \vartheta|a|^2 a) \\
&\quad = e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}(a\Psi - a\mathcal{L}_\varepsilon\Psi) \\
&\qquad + e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}(a\mathcal{L}_\varepsilon\Psi - \vartheta a|a|^2)) =: \theta_3 + \theta_4.
\end{aligned}$$

We start with θ_1:

$$\begin{aligned}
\int_0^T \|\theta_1\|_{\mathcal{H}^\alpha} ds &= \int_0^T \|e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}(9\bar{a}\Phi - \vartheta\bar{a}\mathcal{L}_\varepsilon^{+2}\Phi)ds\|_{\mathcal{H}^\alpha} \\
&= \int_0^T \left(\int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} (1+k^2)^\alpha |e^{-(T-s)\varepsilon^{-2}(1-(\varepsilon k+1)^2)^2}\mathcal{F}(9\bar{a}\Phi - \bar{a}\mathcal{L}_\varepsilon^{+2}\Phi)|^2 dk\right)^{\frac{1}{2}} ds \\
&\leq \int_0^T \left(\int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} (1+k^2)^\alpha e^{-2(T-s)k^2} |\mathcal{F}(9\bar{a}\Phi - \bar{a}\mathcal{L}_\varepsilon^{+2}\Phi)|^2 dk\right)^{\frac{1}{2}} ds,
\end{aligned}$$

where the last step is justified because

$$0 \leq e^{-(t-s)\varepsilon^{-2}(1-(\varepsilon k+1)^2)^2} = e^{-(t-s)k^2(4+4\varepsilon k+\varepsilon^2 k^2)}$$

and

$$p(k) := 4 + 4\varepsilon k + \varepsilon^2 k^2$$

satisfies $p(k) \geq 4$ for $k > 0$ and is monotone for $k \in [-\varepsilon^{-1}, 0]$. So for these k we get

$$1 = 4 - 4\varepsilon\varepsilon^{-1} + \varepsilon^2\varepsilon^{-2} \leq 4 + 4\varepsilon k + \varepsilon^2 k^2 \leq (4 + 4\varepsilon \cdot 0 + \varepsilon^2 \cdot 0) = 4,$$

which means $p(k) \geq 1$ for all $k \in [-\varepsilon^{-1}, \varepsilon^{-1}]$.

Because for all $\gamma > 0$ there exists a constant C such that the inequality

$$(1+x)^\gamma e^{-x} \leq C \tag{5.45}$$

holds for all $x \geq -1$ we can further bound θ_1 by

$$\begin{aligned}
\int_0^T \|\theta_1\|_{\mathcal{H}^\alpha}\, ds &\leq \int_0^T \Bigg(\int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} \frac{(1+k^2)^\alpha}{(1+(T-s)k^2)^\alpha} (1+(T-s)k^2)^\alpha \\
&\qquad\qquad \times e^{-2(T-s)k^2} |\mathcal{F}(9\bar{a}\Phi - \bar{a}\mathcal{L}_\varepsilon^{+2}\Phi)|^2 dk \Bigg)^{\frac{1}{2}} ds \\
&\leq C \int_0^T (T-s)^{-\alpha/2} \Bigg(\int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} |\mathcal{F}(9\bar{a}\Phi - \bar{a}\mathcal{L}_\varepsilon^{+2}\Phi)|^2 dk \Bigg)^{\frac{1}{2}} ds \\
&= C \int_0^T (T-s)^{-\alpha/2} \|9\bar{a}\Phi - \bar{a}\mathcal{L}_\varepsilon^{+2}\Phi\|_{L^2} ds \\
&\leq C T_0^{1-\alpha/2} \sup_{s\in[0,T]} \|\bar{a}\|_\infty \|9\Phi - \mathcal{L}_\varepsilon^{+2}\Phi\|_{L^2}.
\end{aligned}$$

So we have

$$\sup_{T\in[0,\tau^*]} \int_0^T \|\theta_1\|_{\mathcal{H}^\alpha}\, ds \leq C T_0^{1-\alpha/2} \sup_{T\in[0,\tau^*]} \|\bar{a}\|_{\mathcal{H}^\alpha} \|9\Phi - \mathcal{L}_\varepsilon^{+2}\Phi\|_{L^2}$$

and the norm $\|9\Phi - \mathcal{L}_\varepsilon^{+2}\Phi\|_{L^2}$ can be bounded by

$$\begin{aligned}
\|9\Phi - \mathcal{L}_\varepsilon^{+2}\Phi\|_{L^2}^2 &= \big\|\mathcal{F}(9\Phi - \mathcal{L}_\varepsilon^{+2}\Phi)\big\|_{L^2}^2 \\
&= \int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} \big|\big(9 - (1-(\varepsilon k+2)^2)^2\big)\mathcal{F}(\Phi)(k)\big|^2 dk \\
&= \int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} (12\varepsilon k + 10(\varepsilon k)^2 + 8(\varepsilon k)^3 + (\varepsilon k)^4)^2 |\mathcal{F}(\Phi)(k)|^2 dk \\
&\leq C\varepsilon^2 \int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} k^2 |\mathcal{F}(\Phi)(k)|^2 dk \\
&\leq C\varepsilon^2 \int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} (1+k^2)^{1-\alpha}(1+k^2)^\alpha |\mathcal{F}(\Phi)(k)|^2 dk \\
&\leq C\varepsilon^{2\alpha} \|\Phi\|_{\mathcal{H}^\alpha}^2.
\end{aligned}$$

For θ_3 we repeat the same steps to get

$$\begin{aligned}\int_0^{\tau^*} \|\theta_3\|_{\mathcal{H}^\alpha}\, ds &\leq CT_0^{\alpha-1/2} \sup_{T\in[0,\tau^*]} \|a\|_\infty \|\Psi - \mathcal{L}_\varepsilon\Psi\|_{\mathcal{H}^\alpha} \\ &\leq CT_0^{\alpha-1/2}\varepsilon^\alpha \sup_{T\in[0,\tau^*]} \|a\|_{\mathcal{H}^\alpha}\|\Psi\|_{H^\alpha}.\end{aligned}$$

Since we already bounded a with (5.32) and (5.12), Φ in (5.39) and Ψ in (5.40) it follows that

$$\int_0^T \|\theta_1\|_{\mathcal{H}^\alpha} + \|\theta_1\|_{\mathcal{H}^\alpha}\, ds = \mathcal{O}(\varepsilon^{\alpha-4\kappa_0}).$$

From Lemma 24 we can write da as

$$da = (-\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}a + 2\varepsilon^{-1}\vartheta a Z_\varepsilon + r_1)dT, \tag{5.46}$$

where r_1 is given by

$$r_1 = \nu a + 2\vartheta P_\Psi(\bar{a}\Phi + a\Psi + a\Psi_c) - P_\Psi(3a|a|^2 - 3aZ_\varepsilon^2) + R_1,$$

with R_1 being the error term from (5.25). The time integral of its $\mathcal{H}^\alpha$ norm is bounded by

$$\begin{aligned}\|r_1\|_{\mathcal{H}^\alpha} \leq CT_0 \sup_{t\in[0,\tau^*]} (\|a\|_{\mathcal{H}^\alpha}(1 + \|\Phi\|_{\mathcal{H}^\alpha} + \|\Psi\|_{\mathcal{H}^\alpha} + |\Psi_c| + \|a\|^2_{\mathcal{H}^\alpha} + |Z_\varepsilon|^2) \\ + \|R_1\|_{\mathcal{H}^\alpha}).\end{aligned}$$

Thus we find

$$\sup_{T\in[0,\tau^*]} \|r_1\|_{\mathcal{H}^\alpha} = \mathcal{O}(\varepsilon^{-5\kappa_0}). \tag{5.47}$$

The Îto differentials of $\bar{a}\Phi$ and $a\Psi$ are given by

$$\begin{aligned}d(\bar{a}\Phi) &= (d\bar{a})\Phi + \bar{a}(d\Phi) + (d\bar{a})(d\Phi) \\ &= (-\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}\bar{a} + \bar{r}_1 + 2\varepsilon^{-1}\vartheta a Z_\varepsilon)\Phi dT - \bar{a}(\varepsilon^{-2}\mathcal{L}_\varepsilon^{+2}\Phi - \varepsilon^{-2}\vartheta a^2 - R_2)dT \end{aligned} \tag{5.48}$$

$$\begin{aligned}d(a\Psi) &= (da)\Phi + a(d\Phi) + (da)(d\Phi) \\ &= (-\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}a + r_1 + 2\varepsilon^{-1}\vartheta a Z_\varepsilon)\Psi dT - a(\varepsilon^{-2}\mathcal{L}_\varepsilon\Psi - 2\varepsilon^{-2}\vartheta|a|^2 - R_3)dT.\end{aligned} \tag{5.49}$$

Define

$$\begin{aligned}r_2 &:= \mathcal{L}_\varepsilon^{+1}\bar{a}\Phi + \bar{a}\mathcal{L}_\varepsilon^{+2}\Phi - \vartheta\bar{a}a^2 - \varepsilon^2(\bar{r}_1 + 2\varepsilon^{-1}\vartheta a Z_\varepsilon)\Phi - \varepsilon^2\bar{a}R_2 \\ r_3 &:= \mathcal{L}_\varepsilon^{+1}a\Psi + a\mathcal{L}_\varepsilon\Psi - \vartheta a|a|^2 - \varepsilon^2(r_1 + 2\varepsilon^{-1}\vartheta a Z_\varepsilon)\Psi - \varepsilon^2 aR_3.\end{aligned}$$

with R_2 and R_3 being the error terms from (5.26) and (5.27). Now we can split θ_2 and θ_4 each into three parts:

$$\begin{aligned}\theta_2 &= e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}(\bar{a}\mathcal{L}_\varepsilon^{+2}\Phi - \vartheta\bar{a}a^2)\\ &= e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}(\mathcal{L}_\varepsilon^{+1}\bar{a})\Phi + e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}r_2\\ &\quad + \varepsilon^2 e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}((\bar{r}_1 + 2\varepsilon^{-1}\vartheta a Z_\varepsilon)\Phi + \bar{a}R_2)\\ \theta_4 &= e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}(\bar{a}\mathcal{L}_\varepsilon\Psi - \vartheta a|a|^2)\\ &= e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}(\mathcal{L}_\varepsilon^{+1}a)\Psi + e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}r_3\\ &\quad + \varepsilon^2 e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}((r_1 + 2\varepsilon^{-1}\vartheta a Z_\varepsilon)\Psi + aR_3).\end{aligned}$$

The first parts can be bounded in the same way as θ_1 by

$$\begin{aligned}&\int_0^T \|e^{-\varepsilon^{-2}(T-s)\mathcal{L}_\varepsilon^{+1}}(\mathcal{L}_\varepsilon^{+1}\bar{a})\Phi\|_{\mathcal{H}^\alpha} + \|e^{-\varepsilon^{-2}(T-s)\mathcal{L}_\varepsilon^{+1}}(\mathcal{L}_\varepsilon^{+1}a)\Psi\|_{\mathcal{H}^\alpha}ds\\ &\le \int_0^T (T-s)^{-\alpha/2}(\|(\mathcal{L}_\varepsilon^{+1}\bar{a})\Phi\|_{L^2} + \|(\mathcal{L}_\varepsilon^{+1}a)\Psi\|_{L^2})ds\\ &\le CT_0^{1-\alpha/2}\sup_{T\in[0,\tau^*]}(\|\Phi\|_\infty + \|\Psi\|_\infty)\|\mathcal{L}_\varepsilon^{+1}a\|_{L^2}\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{L}_\varepsilon^{+1}a\|_{L^2}^2 &= \|\mathcal{F}(\mathcal{L}_\varepsilon^{+1}a)\|_{L^2}^2\\ &= \int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} (4|\varepsilon k|^2 + 4|\varepsilon k|^3 + |\varepsilon k|^4)^2|\mathcal{F}(a)(k)|^2dk\\ &\le C\int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} |\varepsilon k|^4|\mathcal{F}(a)(k)|^2dk\\ &= C\varepsilon^4\int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} (1+k^2)^{2-\alpha}(1+k^2)^\alpha|\mathcal{F}(a)(k)|^2dk \le C\varepsilon^{2\alpha}\|a\|_{\mathcal{H}^\alpha}^2.\end{aligned}$$

For the second terms we use partial integration ($j \in \{2,3\}$):

$$
\begin{aligned}
&\| \int_0^T e^{-(T-s)\mathcal{L}_\varepsilon^{+1}} r_j \, ds \|_{\mathcal{H}^\alpha} \\
&\quad = \| \int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} \int_0^T e^{-(T-s)p(k)k^2} \mathcal{F}(r_j(s)) ds \, e^{ikx} dk \|_{\mathcal{H}^\alpha} \\
&\quad \leq \| \int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} [e^{-(T-s)p(k)k^2} \int_0^s \mathcal{F}(r_j(\tau)) d\tau]_0^T e^{ikx} dk \|_{\mathcal{H}^\alpha} \\
&\qquad + \| \int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} \int_0^T \Big(k^2 e^{-(T-s)p(k)k^2} \int_0^s \mathcal{F}(r_j(\tau)) d\tau \Big) ds \, e^{ikx} dk \|_{\mathcal{H}^\alpha} \\
&\quad = \left(\int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} (1+k^2)^\alpha e^{-2(T-s)p(k)k^2} |\mathcal{F}(\int_0^T r_j(s) ds)|^2 dk \right)^{1/2} \\
&\qquad + \left(\int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} (1+k^2)^\alpha \left(\int_0^T (p(k)k^2)^2 e^{-2(T-s)p(k)k^2} |\mathcal{F}(\int_0^s r_j d\tau)|^2 ds \right)^{1/2} dk \right)^{1/2} \\
&\quad \leq \| \int_0^T r_j(s) ds \|_{\mathcal{H}^\alpha} + C T_0 \int_0^T (T-s)^{-1/2} \| \int_0^s r_j(\tau) d\tau \|_{\mathcal{H}^\alpha} ds,
\end{aligned}
$$

where we used (5.45) and $p(k) \geq 1$ for $k \in [-\varepsilon^{-1}, \varepsilon^{-1}]$ in the last step. Therefore from (5.48) and (5.49) it follows that

$$
\begin{aligned}
\| \int_0^T e^{-\varepsilon^{-2}(T-s)\mathcal{L}_\varepsilon^{+1}} r_2 \, ds \|_{\mathcal{H}^\alpha} &\leq \varepsilon^2 \| \bar{a}\Phi \|_{\mathcal{H}^\alpha} + \varepsilon^2 C T_0^{3/2} \sup_{T \in [0,\tau^*]} \| \bar{a}\Phi \|_{\mathcal{H}^\alpha} \\
\| \int_0^T e^{-\varepsilon^{-2}(T-s)\mathcal{L}_\varepsilon^{+1}} r_3 \, ds \|_{\mathcal{H}^\alpha} &\leq \varepsilon^2 \| a\Psi \|_{\mathcal{H}^\alpha} + \varepsilon^2 C T_0^{3/2} \sup_{T \in [0,\tau^*]} \| a\Psi \|_{\mathcal{H}^\alpha}.
\end{aligned}
$$

The last parts are simply bounded by

$$
\begin{aligned}
&\varepsilon^2 \int_0^T \| e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}} ((r_1 + 2\varepsilon^{-1}\vartheta a Z_\varepsilon)\Phi + aR_3) \|_{\mathcal{H}^\alpha} ds \\
&\qquad \leq \varepsilon^2 T_0 \sup_{T \in [0,\tau^*]} \| (\bar{r}_1 + 2\varepsilon^{-1}\vartheta a Z_\varepsilon)\Phi + \bar{a} R_2 \|_{\mathcal{H}^\alpha} \\
&\qquad \leq C(\varepsilon^2 \|r_1\|_{\mathcal{H}^\alpha} \|\Phi\|_{\mathcal{H}^\alpha} + \varepsilon \|a\|_{\mathcal{H}^\alpha} |Z_\varepsilon| \|\Phi\|_{\mathcal{H}^\alpha} + \varepsilon^2 \|a\|_{\mathcal{H}^\alpha} \|R_2\|_{\mathcal{H}^\alpha}) \\
&\varepsilon^2 \int_0^T \| e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}} ((r_1 + 2\varepsilon^{-1}\vartheta a Z_\varepsilon)\Psi + aR_3) \|_{\mathcal{H}^\alpha} ds \\
&\qquad \leq \varepsilon T_0 \sup_{T \in [0,\tau^*]} \| (r_1 + 2\varepsilon^{-1}\vartheta a Z_\varepsilon)\Psi + aR_3 \|_{\mathcal{H}^\alpha} \\
&\qquad \leq C(\varepsilon^2 \|r_1\|_{\mathcal{H}^\alpha} \|\Psi\|_{\mathcal{H}^\alpha} + \varepsilon \|a\|_{\mathcal{H}^\alpha} |Z_\varepsilon| \|\Psi\|_{\mathcal{H}^\alpha} + \varepsilon^2 \|a\|_{\mathcal{H}^\alpha} \|R_3\|_{\mathcal{H}^\alpha}).
\end{aligned}
$$

Now we set

$$\begin{aligned} R_5 &:= \theta_1 + (\theta_2 - e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}} r_2) \\ R_6 &:= e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}} r_2 \\ R_7 &:= \theta_3 + (\theta_4 - e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}} r_3) \\ R_8 &:= e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}} r_3, \end{aligned}$$

which finishes the proof for Equations (5.41) and (5.42).

The Itô differential of $e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}} aZ_\varepsilon$ is given by

$$\begin{aligned} &d(e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}} aZ_\varepsilon) \\ &\quad = \varepsilon^{-2}\mathcal{L}_\varepsilon^{+1} e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}(aZ_\varepsilon)ds + e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}((da)Z_\varepsilon + a(dZ_\varepsilon)) \\ &\quad = \varepsilon^{-2}\mathcal{L}_\varepsilon^{+1} e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}(aZ_\varepsilon)ds + e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}(-\varepsilon^{-2}(\mathcal{L}_\varepsilon^{+1}a)Z_\varepsilon + \varepsilon^{-1}aZ_\varepsilon^2)ds \\ &\qquad + e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}(r_1 Z_\varepsilon - \varepsilon^{-2}aZ_\varepsilon)ds + e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}(\varepsilon^{-1}\sigma a)d\beta \\ &\quad = e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}(\varepsilon^{-1}aZ_\varepsilon^2 + r_1 Z_\varepsilon - \varepsilon^{-2}aZ_\varepsilon)ds + e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}(\varepsilon^{-1}\sigma a)d\beta, \end{aligned}$$

where we used (5.46). Since $\mathcal{L}_\varepsilon(aZ_\varepsilon) = (\mathcal{L}_\varepsilon a)Z_\varepsilon$ because Z_ε is constant in space we have

$$\begin{aligned} e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}} aZ_\varepsilon = \int_0^T & e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}(\varepsilon^{-1}aZ_\varepsilon^2 + r_1 Z_\varepsilon - \varepsilon^{-2}aZ_\varepsilon)ds \\ &+ \int_0^T e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}(\varepsilon^{-1}\sigma a)d\beta. \end{aligned}$$

Therefore,

$$\begin{aligned} &\|\int_0^T e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}(\varepsilon^{-1}aZ_\varepsilon - aZ_\varepsilon^2)ds - \int_0^T e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}\sigma a d\beta\|_{\mathcal{H}^\alpha} \\ &\quad \le \varepsilon\|e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}} aZ_\varepsilon\|_{\mathcal{H}^\alpha} + \varepsilon\|\int_0^T e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}} r_1 Z_\varepsilon\, ds\|_{\mathcal{H}^\alpha} \\ &\quad \le C\varepsilon(\|aZ_\varepsilon\|_{\mathcal{H}^\alpha} + T_0\|r_1 Z_\varepsilon\|_{\mathcal{H}^\alpha}) \le C\varepsilon(1+T_0)|Z_\varepsilon|(\|a\|_{\mathcal{H}^\alpha} + \|r_1\|_{\mathcal{H}^\alpha}). \end{aligned}$$

If we repeat these steps for $a\Psi_c$, then we derive

$$\begin{aligned} &d(e^{-(t-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}} a\Psi_c) \\ &\quad = e^{-(t-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}(\varepsilon^{-1}aZ_\varepsilon\Psi_c + r_1\Psi_c - \varepsilon^{-2}a\Psi_c + \varepsilon^{-2}\vartheta aZ_\varepsilon^2 + aR_4)ds \end{aligned}$$

with R_4 from (5.28) and

$$\begin{aligned} &\|\int_0^T e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}(a\Psi_c - \vartheta aZ_\varepsilon^2)ds\|_{\mathcal{H}^\alpha} \\ &\quad \le \varepsilon^2\|e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}} a\Psi_c\|_{\mathcal{H}^\alpha} + \varepsilon\|\int_0^T e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}(a\Psi_c Z_\varepsilon + \varepsilon r_1\Psi_c + \varepsilon aR_4)ds\|_{\mathcal{H}^\alpha} \\ &\quad \le C\varepsilon(\|a\Psi_c\|_{\mathcal{H}^\alpha} + T_0\|a\Psi_c Z_\varepsilon + \varepsilon r_1\Psi_c + \varepsilon aR_4\|_{\mathcal{H}^\alpha}) \\ &\quad \le C\varepsilon(1+T_0)(\|a\|_{\mathcal{H}^\alpha}|\Psi_c|(1+|Z_\varepsilon|) + \|r_1\|_{\mathcal{H}^\alpha}|\Psi_c| + \varepsilon\|a\|_{\mathcal{H}^\alpha}\|R_4\|_{\mathcal{H}^\alpha}). \end{aligned}$$

This proves (5.43) and (5.44). $\square$

5.2.6 Exchange of the semigroup

The next Lemma gives the error resulting from the exchange of the semigroups generated by the operators $-\mathcal{L}^{+1}$ and $4\partial_x^2$.

Lemma 26. *Let $X(t) \in \mathcal{H}^\alpha$, $\alpha < 1$ with $\operatorname{supp} \mathcal{F}(A(t)) \subset [-\frac{3}{4}\varepsilon^{-1}, \frac{3}{4}\varepsilon^{-1}]$ for all $t \in [0, T_0]$ and $\mathcal{L}^{+1} := (1 + (\varepsilon\partial_x + i)^2)^2$ as defined in Definition 20 then*

$$\sup_{T\in[0,T_0]} \int_0^T \|(e^{-\varepsilon^{-2}(T-s)\mathcal{L}_\varepsilon^{+1}} - e^{4(T-s)\partial_x^2})X(s)\|_{\mathcal{H}^\alpha} ds \leq C\varepsilon^\alpha \sup_{T\in[0,T_0]} \|X\|_{\mathcal{H}^\alpha}, \tag{5.50}$$

where C is a positive constant.

Proof. For any time $T > 0$ we write

$$\begin{aligned}
\theta &:= \|(e^{4(T-s)\partial_x^2} - e^{-\varepsilon^{-2}(T-s)\mathcal{L}_\varepsilon^{+1}})X(s)\|_{\mathcal{H}^\alpha} \\
&= \Big(\int_{\mathbb{R}} (1+k^2)^\alpha (e^{-4(T-s)k^2} - e^{-(T-s)(4k^2+4\varepsilon k^3+\varepsilon^2k^4)})^2 |\mathcal{F}(X(s))|^2 dk\Big)^{1/2} \\
&= \Big(\int_{-\frac{3}{4}\varepsilon^{-1}}^{\frac{3}{4}\varepsilon^{-1}} (1+k^2)^\alpha e^{-8(T-s)k^2} (1 - e^{-(T-s)(4\varepsilon k^3+\varepsilon^2k^4)})^2 |\mathcal{F}(X(s))|^2 dk\Big)^{1/2}.
\end{aligned}$$

In the case of $|k| \leq \frac{3}{4}\varepsilon^{-1}$ the polynomial $q(k) := -(4k^3 + \varepsilon k^4)$ is bounded by

$$q(k) \leq 4|k|k^2 - \varepsilon k^4 \leq 4|k|k^2 \leq 3\varepsilon^{-1}k^2.$$

Also

$$q(k)^2 \leq (\varepsilon|k| + 4)^2 k^6 \leq (5k^3)^2.$$

Now we use these bounds on $q(k)$ in combination with the following inequality, which is a direct result of the intermediate value theorem:

$$|1 - e^x| \leq |x| \max\{1, e^x\}.$$

For $T-s>0$ we obtain

$$\begin{aligned}
\theta^2 &\le \int_{-\frac{3}{4}\varepsilon^{-1}}^{\frac{3}{4}\varepsilon^{-1}} (1+k^2)^\alpha e^{-8(T-s)k^2}(1-e^{(T-s)\varepsilon q(k)})^2 |\mathcal{F}(X(s))|^2 dk \\
&\le \int_{-\frac{3}{4}\varepsilon^{-1}}^{\frac{3}{4}\varepsilon^{-1}} (1+k^2)^\alpha e^{-8(T-s)k^2}((T-s)\varepsilon q(k))^2 \max\{1, e^{2(T-s)\varepsilon q(k)}\} |\mathcal{F}(X(s))|^2 dk \\
&\le \int_{-\frac{3}{4}\varepsilon^{-1}}^{\frac{3}{4}\varepsilon^{-1}} (1+k^2)^\alpha e^{-8(T-s)k^2}((T-s)\varepsilon 5k^3)^2 \max\{1, e^{(T-s)6k^2)}\} |\mathcal{F}(X(s))|^2 dk \\
&= 25 \int_{-\frac{3}{4}\varepsilon^{-1}}^{\frac{3}{4}\varepsilon^{-1}} k^2\varepsilon^2 \frac{(1+k^2)^\alpha}{(1+(T-s)k^2)^\alpha} \big((1+(T-s)k^2)^\alpha e^{-(T-s)k^2}\big) \\
&\qquad\qquad \times \big(e^{-(T-s)k^2}((T-s)k^2)^2\big)\big(e^{-6(T-s)} \max\{1, e^{(T-s)6k^2)}\}\big) |\mathcal{F}(X(s))|^2 dk \\
&\le C(T-s)^{-\alpha} \int_{-\frac{3}{4}\varepsilon^{-1}}^{\frac{3}{4}\varepsilon^{-1}} \varepsilon^{2\alpha} k^{2\alpha} |\mathcal{F}(X(s))|^2 dk \\
&\le C(T-s)^{-\alpha}\varepsilon^{2\alpha} \|X(s)\|^2_{\mathcal{H}^\alpha}
\end{aligned}$$

From integrating over time follows that

$$\sup_{T\in[0,T_0]} \int_0^T \theta \, ds \le C\varepsilon^{2\alpha} T^{1-\alpha/2} \sup_{T\in[0,T_0]} \|X(s)\|_{\mathcal{H}^\alpha} \le C\varepsilon^{2\alpha} T_0^{1-\alpha/2} \sup_{T\in[0,T_0]} \|X(s)\|_{\mathcal{H}^\alpha},$$

which proves (5.50). □

5.2.7 Energy estimate for the amplitude equation

All results from the last sections hold until the stopping time τ^*. By controlling the moments of the solution $A(T)$ to the amplitude equation we can bound the probability of τ^* being smaller than a fixed time, if we also control the distance of $A(T)$ to the dominant part $a(T)$ of $v(T)$.
To get the needed upper bound on $A(T)$ we derive a global growth condition through several estimates of Gronwall-type that relate the moments of $A(T)$ to its initial value.

Lemma 27. *Let A be a mild solution of the SPDE*

$$dA = (4{\partial_x}^2 A + C_1 A - C_2|A|^2 A)dt + \sigma A d\beta$$

with $C_1 \in \mathbb{R}$ and $C_2 \ge 0$ and paths $A_\omega(t) \in C([0,T_0]; L^\infty)$. Let

$$\|A(0)\|_{\mathcal{H}^\alpha} = \mathcal{O}(\varepsilon^{-\kappa}). \tag{5.51}$$

Then

$$\sup_{t\in[0,T_0]} \|A\|_{\mathcal{H}^\alpha} = \mathcal{O}(\varepsilon^{-6\kappa}).$$

Proof. To abbreviate the following calculations we define the scalar product

$$\langle f \,|\, g \rangle := \int_{\mathbb{R}} f(x)\overline{g(x)}dx.$$

Now if $\|A\|_{\mathcal{H}^2} < \infty$ we can write

$$\begin{aligned} d\|A\|_{L^2}^2 &= d\langle A \,|\, A\rangle \\ &= \langle dA \,|\, A\rangle + \langle A \,|\, dA\rangle + \langle dA \,|\, dA\rangle \\ &= 2\,\mathrm{Re}\{\langle dA \,|\, A\rangle\} + \sigma^2\langle A \,|\, A\rangle dt \\ &= 2\,\mathrm{Re}\{\langle 4{\partial_x}^2 A + C_1 A - C_2|A|^2 A \,|\, A\rangle\}dt + \sigma^2\langle A \,|\, A\rangle dt + 2\sigma\langle A \,|\, A\rangle d\beta \\ &= (-8\,\mathrm{Re}\{\langle \partial_x A \,|\, \partial_x A\rangle\} + 2C_1\|A\|_{L^2}^2 - 2C_2\|A\|_{L^4}^4)dt + 2\sigma\|A\|_{L^2}^2 d\beta \\ &\leq (-\|A\|_{\mathcal{H}^1}^2 - C_2\|A\|_{L^4}^4 + C\|A\|_{L^2}^2)dt + 2\sigma\|A\|_{L^2}^2 d\beta, \end{aligned} \tag{5.52}$$

where C is a positive constant. The existence of the H^2-norm and of the H^1, L^2 and L^4 norms from the last step can be derived by standard parabolic regularity theory from the random SPDE for $B := e^{-\sigma\beta(t)}A$:

$$dB = (4{\partial_x}^2 B + (C_1 - \tfrac{1}{2}\sigma^2)B - C_2|A|^2 B)dt.$$

We note that the paths of $|A|^2$ are bounded in time and space. Therefore, because of parabolic regularity we can, for each path ω, show existence of a mild solution $B_\omega(t) \in C([0,T_0];\mathcal{H}^\alpha)$ by Banachs fixed point theorem. This implies

$$B_\omega \in L^\infty([0,T],\mathcal{H}^\alpha) \cap L^4([0,T],L^4).$$

We can then use regularisation through the semigroup: For any time $t > 0$ and $\delta < 1$ we have

$$\begin{aligned} \|B_\omega(t)\|_{\mathcal{H}^{\alpha+\delta}}^2 \leq C &\int_{\mathbb{R}} (1 + t^{-\delta})(1 + t^\delta k^{2(\alpha+\delta)})e^{-2tk^2}|\mathcal{F}(B_\omega)|^2 dk \\ &+ C_\omega \int_0^t \int_{\mathbb{R}} (1 + (t-s)^{-\delta})(1 + (t-s)^\delta k^{2(\alpha+\delta)})e^{-2(t-s)k^2}|\mathcal{F}(B_\omega)|^2 dk\, ds \\ &\leq C_\omega(1 + t^{-\delta} + t + t^{1-\delta}) \sup_{s\in[0,t]} \|B_\omega(s)\|_{\mathcal{H}^\alpha}, \end{aligned}$$

which shows that

$$B_\omega \in L^2([0,T],\mathcal{H}^{\alpha+\delta})$$

for each path B_ω. Using the state of B_ω after an infinitesimal time as new initial value with higher regularity and repeating this process we can expand the regularity to

$$B_\omega \in C^0((0,T],\mathcal{H}^\infty).$$

Thus since $A(t) = e^{\sigma\beta(t)}B(t)$ and $e^{\sigma\beta(t)}$ has continuous paths we get the existence of above norms for $A(t)$.

Taking the derivative of $\|A\|_{L^2}^{2p}$ gives

$$d\|A(t)\|_{L^2}^{2p} = p\|A(t)\|_{L^2}^{2(p-1)} d\|A(t)\|_{L^2}^2 + (p-1)p\|A(t)\|_{L^2}^{2(p-2)}(d\|A(t)\|_{L^2}^2)^2$$

and if we put in (5.52) we get

$$\begin{aligned}\mathbb{E}\|A(t)\|_{L^2}^{2p} &\leq \mathbb{E}\|A(0)\|_{L^2}^{2p} + 2p\sigma\mathbb{E}\int_0^t \|A(s)\|_{L^2}^{2p} d\beta + C_p \int_0^t \mathbb{E}\|A(s)\|_{L^2}^{2p} ds \\ &= \mathbb{E}\|A(0)\|_{L^2}^{2p} + C_p \int_0^t \mathbb{E}\|A(s)\|_{L^2}^{2p} ds, \end{aligned} \tag{5.53}$$

where for the second step we used that integration preserves the martingale property. From using Gronwalls inequality on (5.53) it follows that

$$\mathbb{E}\|A(t)\|_{L^2}^{2p} \leq e^{C_p t}\mathbb{E}\|A(0)\|_{L^2}^{2p} \tag{5.54}$$

and with this we can now bound the moments of $\sup_{0\leq s\leq t}\|A(s)\|_{L^2}$:

$$\begin{aligned}\mathbb{E}\sup_{0\leq s\leq t}\|A(s)\|_{L^2}^{2p} &\leq \mathbb{E}\|A(0)\|_{L^2}^{2p} + C_p\mathbb{E}\int_0^t \|A(s)\|_{L^2}^{2p} ds + C_p\mathbb{E}\sup_{0\leq s\leq t}\int_0^t \|A(s)\|_{L^2}^{2p} d\beta \\ &\leq \mathbb{E}\|A(0)\|_{L^2}^{2p} + C_p\mathbb{E}\int_0^t \|A(s)\|_{L^2}^{2p} ds + C_p\mathbb{E}(\int_0^t \|A(s)\|_{L^2}^{4p} ds)^{1/2} \\ &\leq C_p(\mathbb{E}\|A(0)\|_{L^2}^{2p} + (E\|A(0)\|_{L^2}^{4p})^{1/2}) \end{aligned} \tag{5.55}$$

where we used the Burkholder Davis Gundy inequality (BDG) in the second step and (5.53) together with the Hölder inequality in the last step.

Also from (5.52) it follows that

$$\begin{aligned}\mathbb{E}(\int_0^t \|A(s)\|_{\mathcal{H}^1}^2 ds)^p \leq C_p\mathbb{E}\|A(0)\|_{L^2}^2 + C_p\mathbb{E}(\int_0^t \|A(s)\|_{L^2}^2 ds)^p \\ + C_p\mathbb{E}(\int_0^t \|A(s)\|_{L^2}^2 d\beta)^p\end{aligned}$$

By again using BDG and Hölder and then putting (5.53) into the result we arrive at

$$\begin{aligned}\mathbb{E}(\int_0^t \|A(s)\|_{\mathcal{H}^1}^2 ds)^p &\leq C_p\mathbb{E}\|A(0)\|_{L^2}^{2p} + C_p\mathbb{E}\int_0^t \|A(s)\|_{L^2}^{2p} ds \\ &\leq C_p(1+te^{C_p t})\mathbb{E}\|A(0)\|_{L^2}^{2p}. \end{aligned} \tag{5.56}$$

Next we bound the moments of $\|A\|_\infty$ which we divide into three terms:

$$\begin{aligned}\mathbb{E}\|A(t)\|_\infty^p \leq \mathbb{E}\|e^{-t\partial_x{}^2}A(0)\|_\infty^p + \mathbb{E}\|\int_0^t e^{-(t-s)\partial_x{}^2} C_1A - C_2|A|^2A ds\|_\infty^p \\ + \mathbb{E}\|\int_0^t e^{-(t-s)\partial_x{}^2}\sigma A(s) d\beta\|_\infty^p \\ := I_{1,p} + I_{2,p} + I_{3,p}.\end{aligned}$$

The first term is simply bounded by (A.4) (see Appendix A) and the $\mathcal{H}^\alpha$ norm of the initial value:

$$I_{1,p} = \mathbb{E}\|e^{-t\partial_x{}^2} A(0)\|_\infty^p \leq \mathbb{E}\|A(0)\|_\infty^p \leq \mathbb{E}\|A(0)\|_{\mathcal{H}^\alpha}^p$$

By using the Agmon inequality which is valid on $\mathbb{R}$

$$\|A\|_\infty^2 \leq \|A\|_{L^2}\|A\|_{H^1}, \tag{5.57}$$

we can bound $I_{2,p}$ by

$$\begin{aligned}
I_{2,p} &\leq \mathbb{E}(\int_0^t C_1\|A(s)\|_\infty + C_2\|A(s)\|_\infty^3 ds)^p \\
&\leq C\mathbb{E}(\int_0^t \|A(s)\|_{L^2}^{1/2}\|A(s)\|_{H^1}^{1/2} + \|A(s)\|_{L^2}^{3/2}\|A(s)\|_{H^1}^{3/2} ds)^p \\
&\leq CT_0^{p\frac{3}{2}}\mathbb{E}(\int_0^t \|A(s)\|_{L^2}^2\|A(s)\|_{\mathcal{H}^1}^2 ds)^p \\
&\leq C_p\mathbb{E}\sup_{0\leq s\leq t}\|A(s)\|_{L^2}^{2p}(\int_0^t \|A(s)\|_{\mathcal{H}^1}^2 ds)^p \\
&\leq C_p(\mathbb{E}\sup_{0\leq s\leq t}\|A(s)\|_{L^2}^{4p})^{1/2}(\mathbb{E}(\int_0^t \|A(s)\|_{\mathcal{H}^1}^2 ds)^{2p})^{1/2} \\
&\leq C_p(\mathbb{E}\|A(0)\|_{L^2}^{2p} + (\mathbb{E}\|A(0)\|_{L^2}^{2p})^{1/2}),
\end{aligned}$$

where we used (5.55) and (5.56) in the last step.

For $I_{3,p}$ we need a generalisation of BDG which is proved in [HS01] and states that for a contractive semigroup T on a separable Hilbert space H and a Q-Wiener process the following bound holds for all progressively measurable Hilbert-Schmidt operator valued processes ψ:

$$\mathbb{E}\sup_{0\leq t\leq T}\|\int_0^t T(s)\psi(s)dW(s)\|^p \leq C_p\mathbb{E}(\int_0^T \|\psi(s)\|_{\mathscr{L}_2}^2 ds)^{p/2} \tag{5.58}$$

where $\mathscr{L}_2$ is the standard norm for the space of Hilbert Schmidt operators from the range of Q to H. In our case the involved spaces are rather trivial and our Wiener process is just a Brownian motion, so $Q = id : \mathbb{R} \to \mathbb{R}$ and $\psi(s)x = \sigma A(s)\cdot x$. Thus

$$\begin{aligned}
I_{3,p} &\leq \mathbb{E}\sup_{0\leq s\leq t}\|\int_0^t e^{-(t-s)\partial_x{}^2}\sigma A(s)d\beta\|_{H^1}^p \\
&\leq C_p\mathbb{E}(\int_0^t \|\sigma A(s)\|_{\mathcal{H}^1}^2 ds)^{p/2} \\
&\leq C_p\mathbb{E}\|\sigma A(0)\|_{L^2}^p,
\end{aligned}$$

where we used (5.56) in the last step. Now we can bound the moments of the $\mathcal{H}^\alpha$ norm by

$$\begin{aligned}
d\|A(t)\|^2_{\mathcal{H}^\alpha} &= d\langle D^\alpha A|D^\alpha A\rangle \\
&= 2\,\mathrm{Re}\{\langle D^\alpha(dA)|D^\alpha A\rangle\} + \langle D^\alpha(dA)|D^\alpha(dA)\rangle \\
&= 2\langle D^{2\alpha}A|(dA)\rangle + \sigma^2\langle D^\alpha A|D^\alpha A\rangle dt \\
&= 2\langle D^{2\alpha}A| -4D^2A + C_1A - C_2|A|^2A\rangle dt + 2\sigma\langle D^{2\alpha}A|A\rangle d\beta + \sigma^2\|A\|^2_{H^\alpha} \\
&\le -8\|A\|^2_{H^{1+\alpha}} + (2C_1+\sigma^2)\|A\|^2_{H^\alpha} + 2C_2\|A\|_{H^{2\alpha}}\|A\|^2_\infty\|A\|_{L^2} + 2\sigma\|A\|^2_{\mathcal{H}^\alpha}d\beta
\end{aligned}$$

where we used the Hölder inequality in the last step. By Young's inequality as well as $\|f\|_{\mathcal{H}^{1+\alpha}} \ge \|f\|_{\mathcal{H}^{2\alpha}}$ because of $\alpha \le 1$ we get

$$d\|A(t)\|^2_{\mathcal{H}^\alpha} \le C(\|A\|^2_{\mathcal{H}^\alpha} + \|A\|^6_\infty + \|A\|^6_{L^2})dt + 2\sigma\|A\|^2_{\mathcal{H}^\alpha}d\beta.$$

Therefore

$$\begin{aligned}
\mathbb{E}\|A(t)\|^{2p}_{\mathcal{H}^\alpha} - \mathbb{E}\|A(0)\|^{2p}_{\mathcal{H}^\alpha} &= \mathbb{E}\int_0^t p\|A\|^{2(p-1)}_{\mathcal{H}^\alpha} d\|A(t)\|^2_{\mathcal{H}^\alpha} \\
&\le C_p\int_0^t \mathbb{E}\|A\|^{2p}_{\mathcal{H}^\alpha} + \mathbb{E}(\|A\|^{6p}_\infty + \|A\|^{6p}_{L^2})ds \\
&\le C_p\int_0^t \mathbb{E}\|A\|^{2p}_{\mathcal{H}^\alpha} + 2\mathbb{E}\|A(0)\|^{6p}_{L^2}ds
\end{aligned}$$

and by Gronwalls Lemma

$$\mathbb{E}\|A(t)\|^{2p}_{\mathcal{H}^\alpha} \le e^{C_pt}\mathbb{E}\|A(0)\|^{2p}_{\mathcal{H}^\alpha} + C_pte^{C_pt}\mathbb{E}\|A(0)\|^{6p}_{L^2}, \tag{5.59}$$

thus $\|A(t)\|_{\mathcal{H}^\alpha} = \mathcal{O}(\varepsilon^{-6\kappa})$. It is only left to bound the moments of the supremum of the $\mathcal{H}^\alpha$ norm, which we do in the same way as we bounded the supremum of the L^2 norm in (5.55). Using BDG we find:

$$\begin{aligned}
\mathbb{E}\sup_{0\le s\le t}\|A(t)\|^{2p}_{\mathcal{H}^\alpha} &\le \mathbb{E}\|A(0)\|^{2p}_{\mathcal{H}^\alpha} + C_p\int_0^t \mathbb{E}\|A\|^{2p}_{\mathcal{H}^\alpha} + \mathbb{E}(\|A\|^{6p}_\infty + \|A\|^{6p}_{L^2})ds \\
&\qquad + \mathbb{E}\sup_{0\le s\le t}\int_0^t \|A\|^{2p}_{\mathcal{H}^\alpha}d\beta \\
&\le C_p(\mathbb{E}\|A(0)\|^{2p}_{\mathcal{H}^\alpha} + \mathbb{E}\|A(0)\|^{6p}_{L^2} + \mathbb{E}(\int_0^t \|A\|^{4p}_{\mathcal{H}^\alpha}ds)^{1/2}) \\
&\le C_p(\mathbb{E}\|A(0)\|^{2p}_{\mathcal{H}^\alpha} + \mathbb{E}\|A(0)\|^{6p}_{L^2} + (t\mathbb{E}\|A\|^{4p}_{\mathcal{H}^\alpha})^{1/2}).
\end{aligned}$$

And from (5.59) follows $\sup_{0\le s\le T_0}\|A(t)\|_{\mathcal{H}^\alpha} = \mathcal{O}(\varepsilon^{-6\kappa})$. □

5.2.8 Averaging Lemma

We need one last step before we can bound $\|A(T) - a(T)\|_{\mathcal{H}^\alpha}$ and in succession show that the probability of $\tau^* < T_0$ is small. One term in the SDE for $a(T)$ has still not been matched to the amplitude equation. We make up for this now.

Lemma 28. *With a and Z_ε as defined in* (5.24) *and* (5.6) *the approximation*

$$\sup_{T\in[0,\tau^*]} \|\int_0^T e^{(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}(aZ^2 - \tfrac{1}{2}\sigma^2 a)ds\|_{\mathcal{H}^\alpha} = \mathcal{O}(\varepsilon^{1-6\kappa_0}) \tag{5.60}$$

holds.

Proof. This bound is established in the same way as equations (5.43) and (5.44) from Lemma 26 with the help of the generalised BDG from the last Lemma. Using Itô formula and (5.13) the time derivative of Z_ε^2 is given by

$$d(Z_\varepsilon^2) = 2(dZ_\varepsilon)Z_\varepsilon + (dZ_\varepsilon)^2 = 2\varepsilon^{-1}Z_\varepsilon^2 dT + 2\varepsilon^{-1}\sigma Z_\varepsilon d\tilde{\beta} + \varepsilon^{-2}\sigma^2 dT. \tag{5.61}$$

Putting this into the time derivative of $e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}aZ_\varepsilon^2$ yields

$$\begin{aligned} d(e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}aZ_\varepsilon^2) &= \varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}(T-s)}aZ_\varepsilon^2 \\ &\quad + e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}(T-s)}((da)Z_\varepsilon^2 + d(Z_\varepsilon^2) + (da)d(Z_\varepsilon^2)) \\ &= e^{\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}(T-s)}(2\varepsilon^{-1}\vartheta aZ_\varepsilon^3 + r_1 Z_\varepsilon^2)ds \\ &\quad + e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}(T-s)}(2\varepsilon^{-2}aZ_\varepsilon^2 ds + 2a\varepsilon^{-1}\sigma Z_\varepsilon d\tilde{\beta}) \\ &\quad + e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}(T-s)}\varepsilon^{-2}a\sigma^2 ds. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\sup_{T\in[0,\tau^*]} \|\int_0^T e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}(aZ_\varepsilon^2 - a\sigma^2)ds\|_{\mathcal{H}^\alpha} \\ &\quad\le \varepsilon^2 C \sup_{T\in[0,\tau^*]} \|a(T)Z_\varepsilon(T)^2\|_{\mathcal{H}^\alpha} \\ &\qquad + \varepsilon C \sup_{T\in[0,\tau^*]} \|\int_0^T e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}\vartheta aZ_\varepsilon^3 + \varepsilon r_1 Z_\varepsilon^2\, ds\|_{\mathcal{H}^\alpha} \\ &\qquad + \varepsilon C \sup_{T\in[0,\tau^*]} \|\int_0^T e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}a(s\wedge\tau^*)\sigma Z_\varepsilon\, d\tilde{\beta}\|_{\mathcal{H}^\alpha} \\ &\quad\le \varepsilon^2 C \sup_{T\in[0,\tau^*]} (\|a\|_{\mathcal{H}^\alpha}|Z_\varepsilon|^2) + \varepsilon C T_0 \sup_{T\in[0,\tau^*]} (\vartheta\|a(T)\|_{\mathcal{H}^\alpha}|Z_\varepsilon|^3 + \varepsilon\|r_1(T)\|_{\mathcal{H}^\alpha}|Z_\varepsilon|^2) \\ &\qquad + \varepsilon C\sigma \sup_{T\in[0,T_0]} \|\int_0^T e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}a(s\wedge\tau^*)\sigma Z_\varepsilon\, d\tilde{\beta}\|_{\mathcal{H}^\alpha}. \end{aligned}$$

Using the generalised BDG (5.58) we can bound the last term by

$$\begin{aligned} \mathbb{E}\sup_{T\in[0,T_0]} \|\int_0^T e^{-(T-s)\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}a(s\wedge\tau^*)Z_\varepsilon\, d\tilde{\beta}\|_{\mathcal{H}^\alpha}^p &\le C_p\mathbb{E}(\int_0^{T_0}\|a(s\wedge\tau^*)Z_\varepsilon\|_{\mathcal{H}^\alpha}^2\, ds)^{p/2} \\ &\le C_p T_0^{p/2}(\mathbb{E}\sup_{T\in[0,\tau^*]}\|a\|_{\mathcal{H}^\alpha}^2|Z_\varepsilon|^2)^{p/2}. \end{aligned}$$

By definition of τ^* we have $\sup_{T\in[0,\tau^*]}\|a(T)\|_{\mathcal{H}^\alpha} = \mathcal{O}(\varepsilon^{-\kappa_0})$ and from (5.47) and (5.33) we know that $\sup_{T\in[0,\tau^*]}\|r_1(T)\|_{\mathcal{H}^\alpha} = \mathcal{O}(\varepsilon^{-5\kappa_0})$ as well as $\sup_{T\in[0,\tau^*]}|Z_\varepsilon(T)|^3 = \mathcal{O}(\varepsilon^{-\kappa_0})$. Putting this this into our calculation results in (5.61). □

5.2.9 Approximation of the dominant frequency spectrum

With Lemma 25, 26 and 28 we have narrowed the difference between $a(T)$ and $A(T)$ down to an error R on the right-hand side of their SDEs. In a similar way as we established the bound (5.51) for $A(T)$ we are able to bound $\|A(T)-a(T)\|_{\mathcal{H}^\alpha}$. But the processes need to be altered in a way such that the error R is known even after the time τ^* and that the resulting bound is still valid for $\|A(T)-a(T)\|_{\mathcal{H}^\alpha}$.

Lemma 29. *Let $A(t)$ and $a(t)$ be the processes defined by* (5.3) *and* (5.24), *then*

$$\sup_{T\in[0,\tau^*]} \|A(T)-a(T)\|_{\mathcal{H}^\alpha} = \mathcal{O}(\varepsilon^{\alpha-24\kappa_0}).$$

Proof. From Lemma 24 we know that $a(t)$ is the mild solution of (5.25) and hence can be written as

$$\begin{aligned} a(T) &= \int_0^T e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}(T-s)}\Big(\nu a + 2\vartheta P_\Psi(\bar{a}\Phi + a\Psi + a\Psi_c) \\ &\qquad\qquad - 3P_\Psi(a|a|^2) - 3aZ_\varepsilon^2 + \varepsilon^{-1}2\vartheta a Z_\varepsilon + R_1\Big)ds \\ &= \int_0^T e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}(T-s)}(\nu a + 2\vartheta^2|a|^2 a(\tfrac{1}{9}+2-3) + (2\vartheta^2+4\vartheta^2-3)aZ_\varepsilon^2)ds \\ &\qquad + \int_0^T e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}(T-s)}\sigma a\, d\beta(s) + R_{11} \end{aligned}$$

with the following error term R_{11}:

$$\begin{aligned} R_{11} :=& \int_0^T e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}(T-s)}(1-P_\Psi)(\bar{a}\Phi + a\Psi + a\Psi_c - 3a|a|^2)ds \\ &- \tfrac{2}{9}\vartheta \int_0^T e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}(T-s)}(9\bar{a}\Phi - \vartheta|a|^2 a)ds \\ &- 2\vartheta \int_0^T e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}(T-s)}(a\Psi - 2\vartheta|a|^2 a)ds \\ &- 2\vartheta \int_0^T e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}(T-s)}(a\Psi_c - \vartheta a Z_\varepsilon^2)ds \\ &- \int_0^T e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}(T-s)}(aZ_\varepsilon^2 - \tfrac{1}{2}\sigma a)ds \\ &- \int_0^T e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}(T-s)}(\varepsilon^{-1}2\vartheta a Z_\varepsilon - 4\vartheta^2 a Z_\varepsilon^2)ds + \int_0^T e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}}\sigma a d\beta. \end{aligned}$$

The first integral is bounded by

$$\begin{aligned}
&\int_0^T \|e^{-\varepsilon^{-2}\mathcal{L}_\varepsilon^{+1}(T-s)}(1-P_\Psi)(\bar{a}\Phi + a\Psi + a\Psi_c - 3a|a|^2)\|_{\mathcal{H}^\alpha} ds \\
&\quad \le 2\int_0^T \Big(\int_{1/4\varepsilon^{-1}}^{3/4\varepsilon^{-1}} (1+k^2)^\alpha e^{-2(T-s)k^2} |\mathcal{F}(\bar{a}\Phi + a\Psi + a\Psi_c - 3a|a|^2)|^2 dk\Big)^{1/2} ds \\
&\quad \le 2\varepsilon^\alpha \int_0^T (T-s)^{-\alpha/2} e^{-\frac{1}{16}(T-s)\varepsilon^{-2}} ((T-s)\varepsilon^{-2})^{\alpha/2} \|\bar{a}\Phi + a\Psi + a\Psi_c - 3a|a|^2\|_{\mathcal{H}^\alpha} ds \\
&\quad \le C\varepsilon^\alpha T_0^{1-\alpha/2} \sup_{T\in[0,T_0]} \|\bar{a}\Phi + a\Psi + a\Psi_c - 3a|a|^2\|_{\mathcal{H}^\alpha}
\end{aligned}$$

and (5.39), (5.40) and (5.16) which bound Φ, Ψ and Ψ_c respectively. All other terms are bounded by Lemma 25. We get

$$\sup_{T\in[0,\tau^*]} \|R_{11}\|_{\mathcal{H}^\alpha} = \mathcal{O}(\varepsilon^{\alpha-10\kappa_0}).$$

Since $\operatorname{supp}(a) \subset [-1/4\varepsilon^{-1}, 1/4\varepsilon^{-1}]$ and therefore $\operatorname{supp}(|a|^2 a) \subset [-3/4\varepsilon^{-1}, 3/4\varepsilon^{-1}]$, we can exchange the operator with Lemma 26 which gives us

$$\begin{aligned}
a(T) = a(0) + \int_0^T e^{4(T-s)\partial_x^2}(\nu a + 3(\tfrac{38}{27}\vartheta^2 - 1)|a|^2 a + 3(\vartheta^2 - \tfrac{1}{2})\sigma^2 a) dT \\
+ \int_0^T e^{4(T-s)\partial_x^2} \sigma a d\beta + R_{11} + R_{12}
\end{aligned} \tag{5.62}$$

with

$$\sup_{T\in[0,\tau^*]} \|R_{12}\|_{\mathcal{H}^\alpha} \le C\varepsilon^\alpha \sup_{T\in[0,\tau^*]} \|C_1 a + C_2|a|^2 a + \sigma a\|_{\mathcal{H}^\alpha} = \mathcal{O}(\varepsilon^{\alpha-3\kappa_0}),$$

where $C_1 = \nu + 3(\vartheta^2 - \frac{1}{2})\sigma^2 \in \mathbb{R}$ and $C_2 = 3(\frac{38}{27}\vartheta^2 - 1) > 0$. We combine the errors into $R_{13} := R_{11} + R_{12}$ which results in the bound

$$\sup_{T\in[0,\tau^*]} \|R_{13}\|_{\mathcal{H}^\alpha} = \mathcal{O}(\varepsilon^{\alpha-10\kappa_0}). \tag{5.63}$$

But because we can control this error only up to the stopping time τ^* we define the process

$$\hat{a}(t) := \begin{cases} a(t) & \text{for } t \le \tau^* \\ e^{\sigma(\beta(t)-\beta(\tau^*))} e^{(4\partial_x^2 - \frac{1}{2}\sigma^2)(t-\tau^*)} a(\tau^*) & \text{for } t > \tau^* \end{cases}$$

which, since $a(t) = \hat{a}(t)$ for all stopping times $t \le \tau^*$, resolves the equation

$$\hat{a}(T) = a(0) + \int_0^{T\wedge\tau^*} e^{4(T-s)\partial_x^2}(C_1 \hat{a} + C_2|\hat{a}|^2\hat{a}) dT + \int_0^T e^{4(T-s)\partial_x^2}\sigma\hat{a}\, d\beta + \hat{R}_{13} \tag{5.64}$$

with error $\hat{R}_{13}$ defined as

$$\hat{R}_{13}(t) := \begin{cases} R_{13}(t) & \text{for } t \le \tau^* \\ R_{13}(\tau^*) & \text{for } t > \tau^*. \end{cases}$$

Because $P_\Psi a = a$ we also know by definition of $\hat{a}$ that $P_\psi \hat{a} = \hat{a}$, so from equation (5.62) follows that

$$\operatorname{supp}(\mathcal{F}(\hat{a} - \hat{R}_{13})) \subset [-\varepsilon^{-1}, \varepsilon^{-1}]$$

and therefore

$$\begin{aligned} \|\hat{a} - \hat{R}_{13}\|_{\mathcal{H}^k} &\le \varepsilon^{-k} \|\hat{a} - \hat{R}_{13}\|_{L^2} \\ &\le \varepsilon^{-k}(1 + |e^{\sigma(\beta(t) - \beta(\tau^*))}|) \sup_{t \in [0,\tau^*]} (\|a(t)\|_{\mathcal{H}^\alpha} + \|R_{13}(t)\|_{\mathcal{H}^\alpha}) < \infty \end{aligned}$$

for all $k \ge 0$. If we use the same approach for $A(t)$ with

$$\hat{A}(t) := \begin{cases} A(t) & \text{for } t \le \tau^* \\ e^{\sigma(\beta(t) - \beta(\tau^*))} e^{(4\partial_x^2 - \frac{1}{2}\sigma^2)(t - \tau^*)} A(\tau^*) & \text{for } t > \tau^* \end{cases}$$

we get $\|\hat{A}(t)\|_{\mathcal{H}^k} \le \|A(t \wedge \tau^*)\|_{\mathcal{H}^k} < \infty$, since we already showed the existence of these norms in Lemma 27, and

$$\hat{A}(T) = a(0) + \int_0^{T \wedge \tau^*} e^{(T-s)\partial_x^2}(C_1 \hat{A} + C_2 |\hat{A}|^2 A) dT + \int_0^T e^{(T-s)\partial_x^2} \sigma \hat{A}\, d\beta.$$

From this and (5.64) it follows that

$$\begin{aligned} \hat{A}(T) - \hat{a}(T) = \int_0^T & e^{(T-s)\partial_x^2} \chi_{[0,\tau^*]} (C_1(\hat{A} - \hat{a}) + C_2(|\hat{A}|^2 \hat{A} - |\hat{a}|^2 \hat{a})) dT \\ & + \int_0^T e^{(T-s)\partial_x^2} \sigma (\hat{A} - \hat{a}) d\beta - \hat{R}_{13}, \end{aligned} \tag{5.65}$$

which means $\hat{A} - \hat{a}_R := A - (\hat{a} - \hat{R}_{13})$ is the mild solution of

$$\begin{aligned} d(\hat{A} - \hat{a}_R) = \big(\partial_x^2(\hat{A} - \hat{a}_R) + \chi_{[0,\tau^*]} C_1(\hat{A} - \hat{a}) - \chi_{[0,\tau^*]} C_2(|\hat{A}|^2 \hat{A} - |\hat{a}|^2 \hat{a})\big) dT \\ + \sigma(\hat{A} - \hat{a}) d\beta. \end{aligned}$$

Similar to the calculation in (5.52) we can now bound the L^2 norm of $\hat{A} - \hat{a}_R$:

$$\begin{aligned} & d\|\hat{A} - \hat{a}_R\|_{L^2}^2 \\ &\quad = 2\operatorname{Re}\{\langle d(\hat{A} - \hat{a}_R), \hat{A} - \hat{a}_R \rangle\} + \langle d(\hat{A} - \hat{a}_R), d(\hat{A} - \hat{a}_R)\rangle \\ &\quad = 2\operatorname{Re}\{\langle \partial_x^2(\hat{A} - \hat{a}_R) + \chi_{[0,\tau^*]} C_1(\hat{A} - \hat{a}) - \chi_{[0,\tau^*]} C_2(|\hat{A}|^2\hat{A} - |\hat{a}|^2\hat{a}), \hat{A} - \hat{a}_R\rangle\} dT \\ &\qquad\qquad + 2\sigma \operatorname{Re}\{\langle \hat{A} - \hat{a}, \hat{A} - \hat{a}_R\rangle\} d\beta + \sigma^2 \|\hat{A} - \hat{a}\|_{L^2} dT \\ &\quad \le -2\|\hat{A} - \hat{a}_R\|_{\mathcal{H}^1}^2 + (4C_1 + \sigma^2)(\|\hat{A} - \hat{a}_R\|_{L^2}^2 + \|\hat{R}_{13}\|_{L^2}^2) \\ &\qquad - 2C_2 \operatorname{Re}\{\langle |\hat{A}|^2 \hat{A} - |\hat{a}_R|^2 \hat{a}_R, \hat{A} - \hat{a}_R\rangle\} + \chi_{[0,\tau^*]} 2C_2 |\langle |\hat{a}_R|^2 \hat{a}_R - |\hat{a}|^2 \hat{a}, A - \hat{a}_R\rangle| \\ &\qquad + 2\sigma \operatorname{Re}\{\langle \hat{A} - \hat{a}, \hat{A} - \hat{a}_R\rangle\} d\beta \end{aligned}$$

The cubic terms are bounded by

$$\begin{aligned}
&\mathrm{Re}\{\langle |\hat{A}|^2\hat{A} - |\hat{a}_R|^2\hat{a}_R, \hat{A} - \hat{a}_R\rangle\} \\
&\quad = \int_{\mathbb{R}} \mathrm{Re}\{(|\hat{A}|^2 A - |\hat{a}_R|^2\hat{a}_R)\overline{(\hat{A} - \hat{a}_R)}\}dx \\
&\quad = \int_{\mathbb{R}} (|\hat{A}|^2 + |\hat{a}_R|^2)|\hat{A} - \hat{a}_R|^2 - \mathrm{Re}\{(\hat{A} - \hat{a}_R)^2\overline{\hat{A}\hat{a}_R}\}dx \\
&\quad \geq \int_{\mathbb{R}} (|\hat{A}|^2 + |\hat{a}_R|^2)|\hat{A} - \hat{a}_R|^2 - \tfrac{1}{2}(|\hat{A}|^2 + |\hat{a}_R|^2)|\hat{A} - \hat{a}_R|^2 dx \geq 0
\end{aligned}$$

and

$$\chi_{[0,\tau^*]}|\langle |\hat{a}_R|^2\hat{a}_R - |\hat{a}|^2\hat{a}, \hat{A} - \hat{a}_R\rangle| \leq \|R_{14}\|_{L^2}^2 + \|\hat{A} - \hat{a}_R\|_{L^2}^2$$

where $R_{14} := \chi_{[0,\tau^*]}(|\hat{a}_R|^2\hat{a}_R - |\hat{a}|^2\hat{a})$. For R_{14} we obtain

$$\begin{aligned}
\|R_{14}\|_{L^2} &= \chi_{[0,\tau^*]}\||\hat{a} - \hat{R}_{13}|^2(\hat{a} - \hat{R}_{13}) - |\hat{a}|^2\hat{a}\|_{L^2}^2 \\
&\leq \chi_{[0,\tau^*]}(3\|\hat{a}^2\hat{R}_{13}\|_{L^2}^2 + 3\|\hat{a}\hat{R}_{13}^2\|_{L^2}^2 + \|\hat{R}_{13}^3\|_{L^2}^2) \\
&\leq \chi_{[0,\tau^*]}C(\|\hat{a}\|_\infty^2\|\hat{R}_{13}\|_{L^2} + \|\hat{a}\|_\infty\|\hat{R}_{13}\|_\infty\|\hat{R}_{13}\|_{L^2}^2 + \|\hat{R}_{13}\|_\infty^2\|\hat{R}_{13}\|_{L^2})
\end{aligned}$$

and since $\sup_{T\in[0,\tau^*]}\|a\|_{\mathcal{H}^\alpha} = \mathcal{O}(\varepsilon^{-\kappa_0})$ and $\sup_{T\in[0,T_0]}\|\hat{R}_{13}\|_{\mathcal{H}^\alpha} = \mathcal{O}(\varepsilon^{\alpha-10\kappa_0})$ we have (since $24\kappa_0 < \alpha$):

$$\sup_{T\in[0,T_0]}\|R_{14}\|_{L^2} = \mathcal{O}(\varepsilon^{\alpha-12\kappa_0}).$$

So if we combine all errors into

$$\begin{aligned}
\rho(t) &:= \|\hat{R}_{13}(t)\|_{\mathcal{H}^\alpha} + \|R_{14}(t)\|_{L^2} \\
\sup_{t\in[0,T_0]}\rho(t) &= \mathcal{O}(\varepsilon^{\alpha-12\kappa_0})
\end{aligned}$$

we find

$$\begin{aligned}
d\|\hat{A} - \hat{a}_R\|_{L^2}^2 \leq -\|\hat{A} - \hat{a}_R\|_{\mathcal{H}^1}^2 dT + C(\|\hat{A} - \hat{a}_R\|_{L^2}^2 + \rho^2)dT& \\
+ 2\sigma\,\mathrm{Re}\{\langle \hat{A} - \hat{a}, \hat{A} - \hat{a}_R\rangle\}d\beta.& \qquad (5.66)
\end{aligned}$$

Taking the derivative of the p-th moment yields

$$\begin{aligned}
&d\|\hat{A} - \hat{a}_R\|_{L^2}^{2p} \\
&\quad = p\|\hat{A} - \hat{a}_R\|_{L^2}^{2(p-1)}d\|\hat{A} - \hat{a}_R\|_{L^2}^2 + \tfrac{1}{2}(p-1)p\|\hat{A} - \hat{a}_R\|_{L^2}^{2(p-2)}(d\|\hat{A} - \hat{a}_R\|_{L^2}^2)^2 \\
&\quad \leq C_p(\|\hat{A} - \hat{a}_R\|_{L^2}^{2(p-1)}(\rho^2 + \|\hat{A} - a\|_{L^2}^2 + \|\hat{A} - \hat{a}_R\|_{L^2}^2)dT \\
&\qquad\qquad + 2p\|\hat{A} - \hat{a}_R\|_{L^2}^{2(p-1)}\mathrm{Re}\{\langle \hat{A} - a, \hat{A} - \hat{a}_R\rangle\}d\beta \\
&\quad \leq 4C_p(\|\hat{A} - \hat{a}_R\|_{L^2}^{2p} + \rho^{2p})dT + 2p\|\hat{A} - \hat{a}_R\|_{L^2}^{2(p-1)}\mathrm{Re}\{\langle \hat{A} - a, \hat{A} - \hat{a}_R\rangle\}d\beta. \qquad (5.67)
\end{aligned}$$

By integrating and taking the expectation value the $d\beta$ term vanishes:

$$\mathbb{E}\|\hat{A}(T)-\hat{a}_R(T)\|_{L^2}^{2p} \leq \mathbb{E}\|\hat{A}(0)-\hat{a}_R(0)\|_{L^2}^{2p} + C_p\int_0^T \mathbb{E}\|\hat{A}-\hat{a}_R\|_{L^2}^{2p} + \mathbb{E}\rho^{2p}\,ds$$

And from Gronwalls inequality follows (with $\hat{A}(0)=\hat{a}(0)=a(0)$)

$$\begin{aligned}\mathbb{E}\|\hat{A}-\hat{a}_R\|_{L^2}^{2p} &\leq C_p\int_0^T e^{(T-s)C_p}(\mathbb{E}\int_0^s \rho(t)^{2p}dt)ds\\ &\leq T_0^2C_pe^{T_0C_p}\mathbb{E}\sup_{T\in[0,T_0]}\rho^{2p}.\end{aligned}\tag{5.68}$$

With this we can bound the moments of $\sup_{T\in[0,T_0]}\|\hat{A}-\hat{a}_R\|_{L^2}$. By integrating (5.67) and taking the expectation value of the supremum we derive

$$\begin{aligned}\mathbb{E}\sup_{T\in[0,T_0]}\|\hat{A}-\hat{a}_R\|_{L^2}^{2p} \leq{}& C_p\mathbb{E}\int_0^{T_0}\|\hat{A}-\hat{a}_R\|_{L^2}^{2p}\,ds + \mathbb{E}\int_0^{T_0}\rho^{2p}\,ds\\ &+C_p\mathbb{E}\sup_{T\in[0,T_0]}\int_0^T\|\hat{A}-\hat{a}_R\|_{L^2}^{2(p-1)}\operatorname{Re}\{\langle\hat{A}-a,\hat{A}-\hat{a}_R\rangle\}d\beta.\end{aligned}$$

Applying (5.68) and BDG results in

$$\begin{aligned}\mathbb{E}\sup_{T\in[0,T_0]}\|\hat{A}-\hat{a}_R\|_{L^2}^{2p} \leq{}& C_p\mathbb{E}\sup_{T\in[0,T_0]}\rho^{2p}\\ &+\mathbb{E}(\int_0^{T_0}\|\hat{A}-\hat{a}_R\|_{L^2}^{4(p-1)}\operatorname{Re}\{\langle\hat{A}-a,\hat{A}-\hat{a}_R\rangle\}^2ds)^{1/2}\\ \leq{}& C_p\mathbb{E}\sup_{T\in[0,T_0]}\rho^{2p}+(\int_0^{T_0}\mathbb{E}(2\|\hat{A}-\hat{a}_R\|_{L^2}^{4p}+\rho^{4p})ds)^{1/2}\\ \leq{}&(C_p+C_{2p}^{1/2}+T_0^{1/2})\mathbb{E}\sup_{T\in[0,T_0]}\rho^{2p}.\end{aligned}$$

We can now bound the moments of the $\mathcal{H}^\alpha$ norm. Taking the derivative yields

$$\begin{aligned}d\|\hat{A}-\hat{a}_R\|_{\mathcal{H}^\alpha}^2&\\ ={}& d\langle D^\alpha(\hat{A}-\hat{a}_R)|D^\alpha(\hat{A}-\hat{a}_R)\rangle\\ ={}& 2\operatorname{Re}\{\langle D^\alpha(\hat{A}-\hat{a}_R)|D^\alpha d(\hat{A}-\hat{a}_R)\rangle\}+\langle D^\alpha d(\hat{A}-\hat{a}_R)|D^\alpha d(\hat{A}-\hat{a}_R)\rangle\\ ={}& 2\operatorname{Re}\{\langle D^{2\alpha}(\hat{A}-\hat{a}_R)|d(\hat{A}-\hat{a}_R)\rangle\}+\langle D^\alpha d(\hat{A}-\hat{a}_R)|D^\alpha d(\hat{A}-\hat{a}_R)\rangle\\ ={}& 2\operatorname{Re}\{\langle D^{2\alpha}(\hat{A}-\hat{a}_R)|-D^2(\hat{A}-\hat{a}_R)+C_1(\hat{A}-\hat{a})-C_2(|\hat{A}|^2\hat{A}-|\hat{a}|^2\hat{a})\rangle\}dt\\ &+2\sigma\operatorname{Re}\{\langle D^{2\alpha}(\hat{A}-\hat{a}_R)|\hat{A}-\hat{a}\rangle\}d\beta+\sigma^2\langle D^\alpha(\hat{A}-\hat{a})|D^\alpha(\hat{A}-\hat{a})\rangle dt\\ \leq{}& -2\|\hat{A}-\hat{a}_R\|_{\mathcal{H}^{1+\alpha}}^2+(2C_1+\sigma^2)(\|\hat{A}-\hat{a}_R\|_{\mathcal{H}^\alpha}^2+\|\hat{A}-\hat{a}\|_{\mathcal{H}^\alpha}^2)\\ &+2(\|\hat{A}-\hat{a}_R\|_{\mathcal{H}^{2\alpha}}^2+C_2^2\|(\hat{A}-\hat{a})(|\hat{A}|^2+|\hat{a}|^2)+\overline{(\hat{A}-\hat{a})}\hat{A}\hat{a}\|_{L^2}^2)\\ &+2\sigma\operatorname{Re}\{\langle D^{2\alpha}(\hat{A}-\hat{a}_R)|\hat{A}-\hat{a}\rangle\}d\beta,\end{aligned}$$

where we used the Hölder inequality in the last step. As $(1+\alpha) \geq 2\alpha$ we obtain

$$\begin{aligned} d\|\hat{A} - \hat{a}_R\|^2_{\mathcal{H}^\alpha} \leq C\big(&\|\hat{A} - \hat{a}_R\|^2_{\mathcal{H}^\alpha} + \|\hat{R}_{13}\|^2_{\mathcal{H}^\alpha} + (\|\hat{A}\|^2_\infty + \|\hat{a}\|^2_\infty)^2\|\hat{A} - \hat{a}_R\|^2_{L^2}\big) \\ &+ 2\sigma \operatorname{Re}\{\langle D^\alpha(\hat{A} - \hat{a}_R)|D^\alpha(\hat{A} - \hat{a})\rangle\}d\beta. \end{aligned}$$

So the derivative of the p-th moment is given by

$$\begin{aligned} d\|\hat{A} - \hat{a}_R\|^2_{\mathcal{H}^\alpha} &= p\|\hat{A} - \hat{a}_R\|^{2(p-1)}_{\mathcal{H}^\alpha}(d\|\hat{A} - \hat{a}_R\|^2_{\mathcal{H}^\alpha}) \\ &\quad + \tfrac{1}{2}(p-1)p\|\hat{A} - \hat{a}_R\|^{2(p-2)}_{\mathcal{H}^\alpha}(d\|\hat{A} - \hat{a}_R\|^2_{\mathcal{H}^\alpha})^2 \\ &\leq C_p\|\hat{A} - \hat{a}_R\|^{2p}_{\mathcal{H}^\alpha} + \|\hat{R}_{13}\|^{2p}_{\mathcal{H}^\alpha} + (\|\hat{A}\|^2_\infty + \|\hat{a}\|^2_\infty)^{2p}\|\hat{A} - \hat{a}_R\|^{2p}_{L^2}dt \\ &\quad + C_p\|\hat{A} - \hat{a}_R\|^{2(p-1)}_{\mathcal{H}^\alpha}2\sigma \operatorname{Re}\{\langle D^\alpha(\hat{A} - \hat{a}_R)|D^\alpha(\hat{A} - \hat{a})\rangle\}d\beta. \end{aligned}$$

Thus

$$\begin{aligned} \mathbb{E}\|\hat{A}(T) - \hat{a}_R(T)\|^{2p}_{\mathcal{H}^\alpha} &\leq \int_0^T C_p\mathbb{E}\|\hat{A} - \hat{a}_R\|^{2p}_{\mathcal{H}^\alpha}ds + C_pT_0\mathbb{E}\sup_{t\in[0,T_0]}\|\hat{R}_{13}\|^{2p}_{\mathcal{H}^\alpha} \\ &\quad + C_pT_0(\mathbb{E}\sup_{t\in[0,T_0]}(\|\hat{A}\|^2_{\mathcal{H}^\alpha} + \|\hat{a}\|^2_{\mathcal{H}^\alpha})^{4p})^{1/2} \\ &\quad \times (\mathbb{E}\sup_{t\in[0,T_0]}(\|\hat{A} - \hat{a}_R\|^{4p}_{L^2} + \|\hat{R}_{13}\|^{4p}_{\mathcal{H}^\alpha}))^{1/2} \\ &:= \int_0^T C_p\mathbb{E}\|\hat{A} - \hat{a}_R\|^{2p}_{\mathcal{H}^\alpha}ds + \rho_2^{2p}. \end{aligned} \tag{5.69}$$

From Lemma 27 we know $\sup_{t\in[0,T_0]}\|A(t)\|_{\mathcal{H}^\alpha} = \mathcal{O}(\varepsilon^{-6\kappa_0})$. Also $\hat{a}$ is bounded by the definition of τ^*, we just bounded the L^2-norm by $\sup_{t\in[0,T_0]}\rho(t)$ and R_{13} is bounded by (5.63), so we have

$$\rho_2 = \mathcal{O}(\varepsilon^{\alpha-24\kappa_0}).$$

By Gronwalls Lemma we get

$$\mathbb{E}\|\hat{A}(T) - \hat{a}_R(T)\|^{2p}_{\mathcal{H}^\alpha} \leq C_pT_0e^{C_pT_0}\rho_2^{2p}. \tag{5.70}$$

Integrating (5.69) again and taking the supremum over time prior to the expectation gives us

$$\begin{aligned} &\mathbb{E}\sup_{t\in[0,T]}\|\hat{A}(T) - \hat{a}_R(T)\|^{2p}_{\mathcal{H}^\alpha} \\ &\leq \int_0^T C_p\mathbb{E}\sup_{s\in[0,T]}\|\hat{A} - \hat{a}_R\|^{2p}_{\mathcal{H}^\alpha}ds + \rho_2^{2p} \\ &\quad + C_p\mathbb{E}\sup_{T\in[0,T_0]}\int_0^T\|\hat{A} - \hat{a}_R\|^{2(p-1)}_{\mathcal{H}^\alpha}\operatorname{Re}\{\langle D^\alpha(\hat{A} - \hat{a}_R)|D^\alpha(\hat{A} - \hat{a})\rangle\}d\beta \\ &\leq \int_0^T C_p\mathbb{E}\sup_{s\in[0,T]}\|\hat{A} - \hat{a}_R\|^{2p}_{\mathcal{H}^\alpha}ds + \rho_2^{2p} + C_p(\int_0^{T_0}\mathbb{E}\|\hat{A} - \hat{a}_R\|^{4p}_{\mathcal{H}^\alpha} + \mathbb{E}\|\hat{R}_{13}\|^{4p}_{\mathcal{H}^\alpha})^{1/2}. \end{aligned}$$

Now from (5.70) and the definition of ρ_2 it follows that

$$\mathbb{E}\sup_{t\in[0,T]}\|\hat{A}(T)-\hat{a}_R(T)\|^{2p}_{\mathcal{H}^\alpha}\le C_p\int_0^T\mathbb{E}\sup_{s\in[0,T]}\|\hat{A}-\hat{a}_R\|^{2p}_{\mathcal{H}^\alpha}ds+C_p\rho_2^{2p}$$

and by using again Gronwalls Lemma, we finally establish

$$\sup_{t\in[0,T_0]}\|\hat{A}(T)-\hat{a}_R(T)\|_{\mathcal{H}^\alpha}\le C_p\rho_2=\mathcal{O}(\varepsilon^{\alpha-24\kappa_0}),$$

which proves the Lemma. □

From Lemma 27 we see that until the stopping time τ^* the projection $a=P_a u$ of the solution to the SPDE (5.1) is well approximated by the solution A of the amplitude equation (5.3). We can now prove Theorem 17.

5.2.10 Removing the error

Proof of Theorem 17. First we show that the probability of τ^* being smaller than T_0 is small. Define the following subset of the probability space Ω:

$$M:=\{\omega\in\Omega:\tau^*(\omega)<T_0\}$$

If $\omega\in M$ then it follows from the definition of τ^* that

$$\varphi(\tau^*(\omega)):=\|P_+v(\tau^*(\omega),x)e^{-i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha}+|P_cv(\tau^*(\omega))|=\varepsilon^{-\kappa_0}.$$

This means moments of $\varphi(\tau^*)$ can be written and bounded below as follows:

$$\mathbb{E}\varphi(\tau^*)^q=\int_{M^c}\varphi(T_0)^q d\mathbb{P}+\int_M(\varepsilon^{-\kappa_0})^q d\mathbb{P}\ge\mathbb{P}(M)\varepsilon^{-q\kappa_0},$$

where $M^c:=\Omega\setminus M$ is the complement set of M. Next we show that $\mathbb{E}\varphi(\tau^*)=\mathcal{O}(\varepsilon^{-\kappa})$. We have

$$\begin{aligned}\mathbb{E}\varphi(\tau^*)\le\mathbb{E}&\sup_{t\in[0,\tau^*]}(\|(P_+v(t))e^{-i\varepsilon^{-1}x}-a(t)\|_{\mathcal{H}^\alpha}+\|a(t)-A(t)\|_{\mathcal{H}^\alpha}+\|A(t)\|_{\mathcal{H}^\alpha})\\&+\mathbb{E}\sup_{t\in[0,\tau^*]}(|P_cv(t)-Z_\varepsilon(t)-e^{-\varepsilon^{-2}t}P_cv(0)|+|Z_\varepsilon(t)|)\\&+\mathbb{E}\sup_{t\in[0,\tau^*]}|P_ce^{-\varepsilon^{-2}t\mathcal{L}}v(0)|.\end{aligned}$$

The first term can be split into

$$\begin{aligned}\|(P_+v(t))e^{-i\varepsilon^{-1}x}-a(t)\|_{\mathcal{H}^\alpha}&=\|((P_+-P_a)v(t))e^{-i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha}\\&\le\|(P_sv(t))e^{-i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha}\\&\le\sum_{j=-3}^{3}\|((P_sv_j)e^{-ij\varepsilon^{-1}x})e^{i(j-1)\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha}.\end{aligned}$$

Since for any function $f \in \mathcal{H}^\alpha$ we get (with $l \in \mathbb{N}$)

$$\begin{aligned}
\|fe^{-ilx}\|^2_{\mathcal{H}^\alpha} &\leq \int_{\mathbb{R}} (1+k^2)^\alpha |\mathcal{F}(f)(k+l\varepsilon^{-1})|^2 dk \\
&= \int_{\mathbb{R}} (1+(k-l\varepsilon^{-1})^2)^\alpha |\mathcal{F}(f)(k)|^2 dk \\
&\leq \int_{\mathbb{R}} 2\varepsilon^{-2\alpha}(1+l^2+k^2)^\alpha |\mathcal{F}(f)(k)|^2 dk \leq 2(1+l^2)\varepsilon^{-2\alpha}\|f\|^2_{\mathcal{H}^\alpha},
\end{aligned} \tag{5.71}$$

the sum above is bounded by

$$\begin{aligned}
&\mathbb{E} \sup_{t\in[0,\tau^*]} \|(P_+ v(t))e^{-i\varepsilon^{-1}x} - a(t)\|_{\mathcal{H}^\alpha} \\
&\quad \leq 2(1+(j-1)^2)\varepsilon^{-\alpha} \sum_{j=-3}^{3} \mathbb{E} \sup_{t\in[0,\tau^*]} \|(P_s v_j)e^{-ij\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} \\
&\quad = \mathcal{O}(\varepsilon^{1-\alpha-4\kappa_0}),
\end{aligned} \tag{5.72}$$

where we used Lemma 23 for last step. Because $24\kappa_0 = 25\kappa$ as per definition (5.11), and we made the assumption $\kappa < \frac{1-\alpha}{5}$, it follows that

$$4\kappa_0 < \frac{4\cdot 25}{5\cdot 24}(1-\alpha) = \frac{20}{24}(1-\alpha) < 1-\alpha$$

and therefore

$$\mathbb{E} \sup_{t\in[0,\tau^*]} \|(P_+ v(t))e^{-i\varepsilon^{-1}x} - a(t)\|_{\mathcal{H}^\alpha} = \mathcal{O}(\varepsilon^{-\kappa}).$$

From Lemma 27, Lemma 29 and again (5.11) it is clear that

$$\mathbb{E} \sup_{t\in[0,\tau^*]} (\|a(t) - A(t)\|_{\mathcal{H}^\alpha} + \|A(t)\|_{\mathcal{H}^\alpha}) = \mathcal{O}(\varepsilon^{-\kappa}).$$

Also from Lemma 23 and the bound (5.33) on Z_ε we know that

$$\mathbb{E} \sup_{t\in[0,\tau^*]} (|P_c(v(t) - e^{-\varepsilon^{-2}t\mathcal{L}}v(0)) - Z_\varepsilon(t)| + |Z_\varepsilon(t)|) = \mathcal{O}(\varepsilon^{-\kappa}).$$

The definition of the operator $\mathcal{L}$ lets us easily calculate the next term:

$$\mathbb{E} \sup_{t\in[0,\tau^*]} |P_c e^{-\varepsilon^{-2}t\mathcal{L}}v(0)| = \mathbb{E} \sup_{t\in[0,\tau^*]} |e^{-\varepsilon^2 t} P_c v(0)| = \mathbb{E}|P_c v(0)| = \mathcal{O}(\varepsilon^{-\kappa}).$$

This proves

$$\mathbb{E}\varphi(\tau^*)^q \leq C_q \varepsilon^{-q\kappa}$$

and thus the probability of M is bounded by

$$\mathbb{P}(M) \leq C_q \varepsilon^{q(\kappa_0-\kappa)}.$$

The last step is to bound the probability of

$$\begin{aligned}\xi_1 := &\|P_+(u(t,\varepsilon^{-1}x) - \varepsilon A(\varepsilon^2 t, x)e^{i\varepsilon^{-1}x})e^{-i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} \\ &+ \|P_-(u(t,\varepsilon^{-1}x) - \varepsilon \bar{A}(\varepsilon^2 t, x)e^{-i\varepsilon^{-1}x})e^{i\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha} \\ &+ |P_c u(t,x) - \varepsilon Z_\varepsilon(\varepsilon^2 t) - P_c e^{-t}u(0)|\end{aligned}$$

and

$$\begin{aligned}\xi_2 := \|u(t,x) - \varepsilon A(\varepsilon^2 t, \varepsilon x)e^{ix} - \varepsilon \bar{A}(\varepsilon^2 t, \varepsilon x)e^{-ix} \\ - \varepsilon Z_\varepsilon(\varepsilon^2 t) - P_c e^{-t}u(0)\|_\infty\end{aligned}$$

being too large (i.e. $\mathbb{P}(\sup_{t\in[0,\varepsilon^{-2}T_0]} \xi_j > \varepsilon^{1+\alpha_j-26\kappa})$), where

$$\begin{aligned}\alpha_1 &:= \alpha \wedge (1-\alpha) \\ \alpha_2 &:= \alpha.\end{aligned}$$

We can split this probability into

$$\begin{aligned}\mathbb{P}(\sup_{t\in[0,\varepsilon^{-2}T_0]} \xi_j > \varepsilon^{1+\alpha_j-26\kappa}) &= \mathbb{P}(M \cap \{\sup_{t\in[0,\varepsilon^{-2}T_0]} \xi_j > \varepsilon^{1+\alpha_j-26\kappa}\}) \\ &\quad + \mathbb{P}(M^c \cap \{\sup_{t\in[0,\varepsilon^{-2}T_0]} \xi_j > \varepsilon^{1+\alpha_j-26\kappa}\}) \\ &=: P_1 + P_2.\end{aligned}$$

P_1 is easily bounded by

$$\mathbb{P}(M \cap \{\sup_{t\in[0,\varepsilon^{-2}T_0]} \xi_j > \varepsilon^{1+\alpha_j-26\kappa}\}) \le \mathbb{P}(M) \le C_q \varepsilon^{q(\kappa_0-\kappa)},$$

so the only thing left to do is to bound P_2. We have

$$\begin{aligned}P_2 &= \mathbb{P}(M^c \cap \{\sup_{t\in[0,\varepsilon^{-2}T_0]} \xi_j > \varepsilon^{1+\alpha_j-26\kappa}\}) \\ &\le \mathbb{P}(\{\sup_{t\in[0,\varepsilon^{-2}\tau^*]} \xi_j > \varepsilon^{1+\alpha_j-26\kappa}\})\end{aligned}$$

and using the Chebychev inequality yields

$$P_2 \le \frac{1}{\varepsilon^{q(1+\alpha_j-26\kappa)}} \mathbb{E}(\sup_{t\in[0,\varepsilon^{-2}\tau^*]} \xi_j^q), \tag{5.73}$$

where q is any positive number. The moments of $\sup_{t\in[0,\varepsilon^{-2}\tau^*]}\xi_j$ are bounded by

$$\begin{aligned}
\sup_{t\in[0,\varepsilon^{-2}\tau^*]}\xi_1^q \le C_q &\sup_{t\in[0,\varepsilon^{-2}T_0]}\|P_+\varepsilon v(\varepsilon^2 t,x)e^{-i\varepsilon^{-1}x}-\varepsilon a(\varepsilon^2 t,x)\|^q_{\mathcal{H}\alpha}\\
&+C_q\sup_{t\in[0,\varepsilon^{-2}\tau^*]]}\|P_-\varepsilon v(\varepsilon^2 t,x)e^{i\varepsilon^{-1}x}-\varepsilon\overline{a}(\varepsilon^2 t,x)\|^q_{\mathcal{H}\alpha}\\
&+C_q\sup_{t\in[0,\varepsilon^{-2}\tau^*]}\|\varepsilon a(\varepsilon^2 t,x)-\varepsilon A(\varepsilon^2 t,x)\|^q_{\mathcal{H}\alpha}\\
&+C_q\sup_{t\in[0,\varepsilon^{-2}\tau^*]}\|\varepsilon\overline{a}(\varepsilon^2 t,x)-\varepsilon\overline{A}(\varepsilon^2 t,x)\|^q_{\mathcal{H}\alpha}\\
&+C_q\sup_{t\in[0,\varepsilon^{-2}\tau^*]}|P_c\varepsilon v(\varepsilon^2 t,x)-\varepsilon Z_\varepsilon(\varepsilon^2 t,x)-e^{-t}P_c\varepsilon v(0)|^q\\
=2C_q\varepsilon &\sup_{t\in[0,\tau^*]}\|P_+v(t,x)e^{-i\varepsilon^{-1}x}-a(t,x)\|^q_{\mathcal{H}\alpha}\\
&+C_q\varepsilon\sup_{t\in[0,\tau^*]}2\|a(t,x)-A(t,x)\|^q_{\mathcal{H}\alpha}\\
&+C_q\varepsilon\sup_{t\in[0,\tau^*]}|P_cv(t,x)-Z_\varepsilon(t,x)-e^{-\varepsilon^{-2}t}P_cv(0)|^q.
\end{aligned}$$

From (5.72), Lemma 29 and Lemma 23 together with $24\kappa_0=25\kappa$ it follows that

$$\mathbb{E}(\xi_1^q)\le C_q\varepsilon^{q(1+(1-\alpha)\wedge\alpha-25\kappa)}=C_q\varepsilon^{q(1+\alpha_1-25\kappa)}.$$

By choosing $q=\max\{\frac{p}{\kappa},\frac{p}{\kappa_0-\kappa}\}$ we get the desired result (5.4).

Now if we look at the supremum norm we can improve the error, because we may, without any penalty,

(1) shift functions in the Fourier space:

$$\begin{aligned}
\|(P_+v(t))e^{-i\varepsilon^{-1}x}-a(t)\|_\infty &= \|((P_+-P_a)v(t))e^{-i\varepsilon^{-1}x}\|_\infty\\
&\le \|(P_sv(t))e^{-i\varepsilon^{-1}x}\|_\infty\\
&\le \sum_{j=-3}^{3}\|(P_sv_j)e^{-ij\varepsilon^{-1}x}\|_\infty\\
&\le \sum_{j=-3}^{3}\|(P_sv_j)e^{-ij\varepsilon^{-1}x}\|_{\mathcal{H}^\alpha}=\mathcal{O}(\varepsilon^{1-25\kappa}),
\end{aligned}$$

(2) rescale functions

$$\|P_+u(t,x)-\varepsilon A(\varepsilon^2 t,\varepsilon x)e^{ix}\|_\infty=\|P_+u(t,\varepsilon^{-1}x)-\varepsilon A(\varepsilon^2 t,x)e^{i\varepsilon^{-1}x}\|_\infty.$$

Therefore ξ_2 is bounded by

$$\begin{aligned}
\xi_2^q\le 2C_q&\sum_{j=-3}^{3}\|(P_sv_j)e^{-ij\varepsilon^{-1}x}\|^q_{\mathcal{H}^\alpha}+\|a(t,x)-A(t,x)\|^q_{\mathcal{H}\alpha}\\
&+|P_cv(t,x)-Z_\varepsilon(t,x)-e^{-\varepsilon^{-2}t}P_cv(0)|^q
\end{aligned}$$

and we get directly from Lemma 23 and Lemma 29 that

$$\sup_{t\in[0,\varepsilon^{-2}\tau^*]} \xi_2^q \leq C_q \varepsilon^{1+\alpha-25\kappa}.$$

Again putting this into (5.73) and choosing $q = \max\{\frac{p}{\kappa}, \frac{p}{\kappa_0-\kappa}\}$ shows (5.5), which finishes the proof of Theorem 17. $\square$

6 Existence of solutions

In this chapter we will complement our results for the approximation of the mild solutions of the Swift-Hohenberg equation (SH) on the bounded and unbounded domain by a short proof of the existence of those solutions up until to a stopping time. Additionally we show that this stopping time is greater than the stopping time τ^* up until which the approximation of $u(t)$ by the solution of the amplitude equation holds, which in turn means the existence of a local solution for (SH) is sufficient for stating our main results. We also make some remarks on how one would prove the existence of solutions to the amplitude equations.

6.1 Bounded domain

The existence of a local solution follows from a fixed point argument.

Lemma 30. *There is a stopping time $\tau > 0$ and a unique stochastic process $u(t)$ with paths $u_\omega \in C^0([0,\tau]; L^\infty([0,2\pi]))$ such that $u(t)$ is a mild solution to equation* (SH) *up until τ, that is, the equation*

$$u(t) = e^{-t(1+\partial_x^2)^2}u(0) + \int_0^t e^{-(t-s)(1+\partial_x^2)^2}[\nu\varepsilon^2 u(s) + \alpha u^2(s) - u^3(s)]ds + \varepsilon \int_0^t e^{-(t-s)}\sigma d\beta(s) \tag{6.1}$$

holds for all $t \leq \tau$.

Proof. We want to use Banachs fixed point theorem on the Picard-like fixed point iteration

$$\Gamma_\omega u := e^{-t(1+\partial_x^2)^2}u_0 + \int_0^t e^{-(t-s)(1+\partial_x^2)^2}[\nu\varepsilon^2 u(s) + \alpha u^2(s) - u^3(s)]ds + \varepsilon \int_0^t e^{-(t-s)}\sigma d\beta_\omega(s) \tag{6.2}$$

which acts on the space

$$\mathcal{B}_u(\omega) := \{u \in C^0([0,T_\omega]; L^\infty([0,2\pi]) : \sup_{t\in[0,T_\omega]} \|u(t) - u_0\|_\infty \leq \tfrac{1}{4}\ ,\ u(0) = u_0\}$$

for each path β_ω of the Brownian motion. For this we need to show that Γ_ω is a contraction and a self mapping. Assume that $u, v \in \mathcal{B}_u(\omega)$ then for all $t \leq T_\omega$ we have

$$\begin{aligned}\|\Gamma(u-v)(t)\|_\infty &\leq T_\omega C \sup_{t\in[0,T_\omega]} (\|u(t)-v(t)\|_\infty + \|u(t)-v(t)\|_\infty^2 + \|u(t)-v(t)\|_\infty^3) \\ &\leq T_\omega C(1+\tfrac{1}{2}+\tfrac{1}{4}) \sup_{t\in[0,T_\omega]} \|u(t)-v(t)\|_\infty,\end{aligned}$$

where the constant C does not depend on T_ω and we used that the semigroup is bounded on L^∞ by Corollary A.3:

$$\begin{aligned}\|\int_0^{T_\omega} e^{-(T_\omega-s)(1+\partial_x^2)^2} u(s) ds\|_\infty &\leq \int_0^{T_\omega} \|e^{-(T_\omega-s)(1+\partial_x^2)^2} u(s)\|_\infty ds \\ &\leq \int_0^{T_\omega} C\|u(s)\|_\infty ds \\ &\leq T_\omega C \sup_{t\in[0,T_\omega]} \|u(t)\|_\infty.\end{aligned}$$

So for T_ω being small enough the map Γ_ω is a contraction. Because of

$$\Gamma_\omega u(t) \overset{L^\infty}{\underset{t\to 0}{\to}} u(0)$$

there exists a time T_ω such that

$$\sup_{t\in[0,T_\omega]} \|\Gamma u(t) - u(0)\|_\infty \leq \tfrac{1}{4},$$

therefore if we chose T_ω small enough Γ_ω is also a self mapping. Applying Banachs fixed point theorem and setting $\tau(\omega) = T_\omega$ we get the desired solution. □

Since the solution $u(t)$ is bounded up to the stopping time τ^* it can be extended at least up until a time greater than τ^*.

Lemma 31. *Let $A(t)$, $t \in [0,T]$ be a stochastic process with continuous paths in $\mathbb{C}$ that solves* (1.5) *in the mild sense with initial value given by* (4.1). *Then Lemma 30 holds with the stopping time τ being greater than the stopping time τ^* defined in* (4.6).

Proof. Define

$$\tau_{max}(\omega) := \sup\{\tau(\omega) \wedge T : u(t,\omega) \text{ solves (SH) up until time } \tau(\omega)\}$$

With the stopping time $\tau^* > 0$ as defined in 4.6 of Chapter 4 we have either $\tau^* < \tau_{max}$ or we get a contradiction to the definition of τ_{max}:

Fix $\omega \in \Omega$ such that $\tau_{max}(\omega) \leq \tau^*(\omega)$. Then we know that

$$\sup_{t\in[0,\tau_{max}(\omega))} \|u(t,\omega)\|_\infty \leq \varepsilon^{-\kappa_0}.$$

This also means that the nonlinearity $f(u) := \varepsilon^2\nu u + \vartheta u^2 - u^3$ stays bounded:

$$\sup_{t\in[0,\tau_{max}(\omega))} \|f(u(t))\|_\infty \leq C \sup_{t\in[0,\tau_{max}(\omega))} (|u|_\infty + |u|_\infty^2 + |u|_\infty^3) \leq C\varepsilon^{-3\kappa_0}.$$

Following the ideas of the proof of Theorem 3.3.4 in [Hen81], when t is bounded away from 0, then for $\alpha < 1$ we can bound the norm

$$\|u\|_\alpha^2 := |\hat{u}_1|^2 + |\hat{u}_{-1}|^2 + \sum_{|k|\neq 1} (1-k^2)^{2\alpha} |\hat{u}_k|^2$$

by

$$\begin{aligned}
\|u\|_\alpha^2 \leq{}& |\hat{u}_1|^2 + |\hat{u}_{-1}|^2 + t^{-\alpha} \sum_{|k|\neq 1} (t(1-k^2)^2)^\alpha e^{-2t(1-k^2)^2} |\hat{u}_k(0)|^2 + |Z(t)|^2 \\
&+ C \sum_{|k|\neq 1} \int_0^{\tau_{max}} t^{-\alpha} (t(1-k^2)^2)^\alpha e^{-2t(1-k^2)^2} |\mathcal{F}(f(u))(k)|^2 \\
\leq{}& \|u(t)\|_{L^2} + C(t^{-\alpha}\|u(0)\|_{L^2} + \sup_{t\in[0,\tau_{max})} |Z(t)|^2 + C\tau_{max}^{1-\alpha} \sup_{t\in[0,\tau_{max})} + \|f(u(t))\|_{L^2}^2) \\
\leq{}& C \sup_{t\in[0,\tau_{max})} (|Z(t)|^2 + \|u(0)\|_\infty^2 + \|u(t)\|_\infty^2 + \|f(u(t))\|_\infty^2) < \infty.
\end{aligned}$$

With this we can show that the limit $u(t) \overset{t\to\tau_{max}}{\longrightarrow} u(\tau_{max})$ exists and thus $u(t)$ can be continuously extended up to time τ_{max}. Suppose $0 < c < \tau < t < \tau_{max}$ and $1/2 < \alpha < 2/3$, then for fixed $\omega \in \Omega$ we have

$$\begin{aligned}
\|u(t) - u(\tau)\|_\infty \leq{}& \|(e^{-(t-\tau)\mathcal{L}} - \mathrm{id})u(\tau)\|_\infty + \|\int_\tau^t e^{-(t-s)\mathcal{L}} f(u)ds\|_\infty + \varepsilon|Z(t) - Z(\tau)| \\
\leq{}& C(e^{-(t-\tau)} - 1)(|\hat{u}_1| + |\hat{u}_{-1}|) \\
&+ \sum_{|k|\neq 1} \int_\tau^t |1-k^2|^{\alpha/2+2(1-\alpha/4)} \frac{(t-s)^{1-\alpha/4}}{(t-s)^{1-\alpha/4}} e^{-(t-s)(1-k^2)^2} |(\mathcal{F}u(0))(k)| ds \\
&+ (t-\tau) \sup_{t\in[0,\tau_{max})} \|f(u(t))\|_\infty + C|t-\tau|^{1/4} \\
\leq{}& C(t-\tau)(|\hat{u}_1| + |\hat{u}_{-1}|) + C((t-\tau) + (t-\tau)^{1/4}) \\
&+ \int_\tau^t (t-s)^{-(1-\alpha/4)} \sum_{|k|\neq 1} |1-k^2|^{-\alpha} (1-k^2)^{3\alpha/2} |(\mathcal{F}u(0))(k)| ds \\
\leq{}& C((t-\tau)(1 + \|u(0)\|_\alpha) + (t-\tau)^{1/4} + (t-\tau)^{\alpha/4} \sup_{t\in[0,\tau_{max})} \|u(t)\|_{3\alpha/2}^2) \\
\leq{}& C((t-\tau) + (t-\tau)^{1/4} + (t-\tau)^{\alpha/4}),
\end{aligned}$$

where we used that Z has Hölder-continuous paths with Hölder-exponent smaller than 1/2. This shows that there is a unique continuous extension $\tilde{u} \in C([0, \tau_{max}], L^\infty)$ of u. If we put $\tilde{u}$ into the right-hand side of (6.1) we get a function that is identical to u for $t \in [0, \tau_{max})$ and, as we also just showed, continuous in τ_{max} so equation (6.1) is fulfilled for $\tilde{u}(\tau_{max})$ and $\tilde{u}$ is a mild solution up until time τ_{max}.
Now we can apply Banachs fixed point theorem as in the proof of Lemma 30 to continue the solution up until a stopping time $\tau > \tau_{max}$ which contradicts the definition of τ_{max}. □

We can do the same for the unbounded domain.

6.2 Unbounded domain

Lemma 32. *There is a stopping time $\tau > 0$ and a unique stochastic process $u(t)$ with paths $u_\omega \in C^0([0, \tau]; \mathcal{H}^\alpha(\mathbb{R}) \oplus \mathbb{R})$ such that $u(t)$ is a mild solution to equation* (5.1) *up until τ, that is, equation* (6.1) *holds for all $t \leq \tau$.*

Proof. As in the proof of Lemma 30 we define a suitable map (6.2) on the appropriate space

$$\mathcal{B}_u(\omega) := \{u \in C^0([0, T_\omega]; \mathcal{H}^\alpha(\mathbb{R}) \oplus \mathbb{R}) : \\ \sup_{t \in [0, T_\omega]} (\|u(t) - u_0\|_{\mathcal{H}^\alpha} + |P_c(u(t) - u_0)|) \leq \tfrac{1}{2}\ ,\ u(0) = u_0\}.$$

It is easily verified that this is again a contraction and a self mapping so we can proceed exactly like we did before. □

Lemma 33. *Let $A(t)$, $t \in [0, T]$ be a stochastic process with continuous paths in $\mathcal{H}^\alpha(\mathbb{R})$ that solves* (5.3) *in the mild sense. Then the stopping time τ from Lemma 32 is greater than the stopping time τ^* defined in* (5.12).

Proof. We define the norm

$$\|u\|_\alpha^2 := \|u\|_{\mathcal{H}^\alpha}^2 + |P_c u|^2.$$

With $\alpha < \beta < \alpha + 1$ and fixed $\omega \in \Omega$ we have

$$\begin{aligned}
\|u(t)\|_\beta^2 \leq |e^{-t} P_c u(0)|^2 + &\int_0^{\tau_{max}} |e^{-(t-s)} P_c f(u(s))|^2 ds + |Z(t)|^2 \\
&+ t^{-(\beta-\alpha)} \int_{\mathbb{R}} (t(1+k^2))^{\beta-\alpha} e^{-2t(1-k^2)^2} (1+k^2)^\alpha |(\mathcal{F} u(0))(k)|^2 dk \\
&+ \int_{\mathbb{R}} \int_0^{\tau_{max}} t^{\beta-\alpha} (t(1+k^2))^{\beta-\alpha} e^{-2t(1-k^2)^2} (1+k^2)^\alpha |\mathcal{F}(f(u(s)))(k)|^2 dt dk \\
&\leq C(1 + t^{-(\beta-\alpha)}) \|u(0)\|_\alpha^2 + C\tau_{max}^{1-(\beta-\alpha)} \sup_{t \in [0, \tau_{max}]} \|f(u(t))\|_\alpha^2 + |Z(t)|^2,
\end{aligned}$$

so $\|u(t)\|_\beta$ is bounded when we bound the time t away from zero. We can then proceed as in the proof for Lemma 31 by showing that $u(t)$ is Hölder continuous in a neighbourhood of τ_{max} which gives us the existence of a continuous extension of $u(t)$. □

6.3 Amplitude equations

Though we assumed existence of solutions $A \in C([0, T_0], \mathbb{C})$ to the amplitude equation (1.5) and $A \in C([0, T_0], \mathcal{H}^\alpha)$ to (1.6), this could also be proven through the following steps which are in part very similar to calculations already done in this work:

1. Use Banachs fixed point theorem for a local solution as in Lemma 30.
2. Get a global growth estimate in terms of the initial value by applying Gronwalls Lemma as done in Corollary 15 and Lemma 27.
3. Use the attained global bound to extend the solution up until time T_0 as in Lemma 31.

7 Possible extensions of results

Let us remark on further extensions of the results presented here. First of all, it is straightforward to consider with the methods used, several different stable quadratic and cubic nonlinearities, for instance those treated in [BHP07], [BH04] and [BM13]. Though the main focus here was a specific nonlinearity that exhibits potential destabilisation via unstable cubic terms which arise from the presence of a quadratic nonlinearity.

We rely on the amplitude equation exhibiting a stable nonlinearity while the original equation may be possibly unstable. Thus we expect that equations with unstable terms that exhibit the same characteristics like $(\partial_x u)^2$ as in the Kardar–Parisi–Zhang equation or $(\partial_x^2 u)^2$ can likely be treated in a similar way.

7.1 Closeness to bifurcation

An interesting new approach was presented in [SPPK11, SPKP13]. While the linear perturbation shifting the bifurcation is mostly of lower order, as the $\nu\varepsilon^2 u$ term in (SH), they consider a perturbation in the differential operator of highest order e.g. $\nu\varepsilon^2\partial_x^4 u$. This seems to lead to similar results, as the lower order perturbations, but the methods of proof have to be different.

A further interesting question is whether it causes major problems when the lower order forcing term does not commute with the linear operator. We conjecture that as long as the non-dominant part of the solution can be approximated by an Ornstein-Uhlenbeck process there is no further impact on the equation for the dominant part of the spectrum since additional terms disappear through averaging.

7.2 Boundary conditions and domains

On the bounded domain different boundary conditions in many cases yield similar results. For instance, in the case of Dirichlet or Neumann conditions for equation (SH) we can consider the Fourier basis given in terms of $\sin(kx)$ or $\cos(kx)$, where only a single mode is changing stability. The amplitude A of the dominant mode

$k = 1$ is in that case only real valued, but apart from that the main result would be the same. The amplitude equation is a one-dimensional ODE containing similar terms as (1.5). Only the constants do change.

We could also treat with similar methods other bounded higher dimensional domains for the underlying SPDE. The main feature for domain and boundary conditions is that the linear operator (in our case $-(1+\partial_x^2)^2$) has a non-negative spectrum and exhibits a basis of eigenfunctions, where the dominating space is given by its finite dimensional kernel. Nevertheless, we then need additional technical conditions, how the non-linearities interact with the eigenfunctions. See [BHP07] for an example of Burgers type in full abstract generality. In order to avoid these technicalities, we consider only our specific example using complex Fourier series.

For higher dimensional unbounded domains the above ansatz fails when the kernel of the linear operator becomes infinite dimensional. So $\mathbb{R}^2$ would not work but $\mathbb{R} \times [0, 2\pi]$ would. We also face the problem that in order to use the embedding of $\mathcal{H}^\alpha$ into L^∞ requires α to be even bigger namely $\alpha > d/2$, where d is the dimension.

7.3 Noise

The assumption that the noise is spatially constant, is easily changed to noise acting on any other non-dominant Fourier-mode or, for the unbounded domain, frequency added to the space $\mathcal{H}^\alpha$ in the same way we constructed the space $\mathcal{H}^\alpha \oplus \mathbb{R}$. Unless frequencies that do not have a distance of order one to the kernel of $\mathcal{L}$ are forced, the main result would be the same. Only constants in (1.5) respectively (1.6) might change.

Noise acting on infinitely many Fourier-modes or frequencies can sometimes be treated by similar methods. However, one needs many assumptions that various infinite series appearing in the calculations do converge. This can be regarded as a regularity assumptions on the noise. Nevertheless there is a key problem with different noise driven frequencies interacting via the nonlinearity. In some cases a fast OU-process inside stochastic integrals needs to be averaged. This can not be averaged directly with strong error estimates as done here in Lemma 13 and Lemma 28, for example. For the bounded domain we know that the main result in principle still remains true, but only weak convergence of approximation to the solution in the limit $\varepsilon \to 0$ is available (see [BHP07]).

A first step towards considering noise acting on whole intervals of the spectrum (in case of the unbounded domain) could be to relax the $\alpha > 1/2$ restriction, as space time white noise for example is only Hölder continuous with exponent smaller than 1/2. If we assume that the conditions on the initial value and existence of solutions are true with the $\mathcal{H}^\alpha$ replaced by the Sobolev-norm

$$\|u\|_{W_{\alpha,p}} := \|\mathcal{F}^{-1}(\zeta_\alpha(k)\mathcal{F}(u))\|_{L^p},$$

then we can drop the lower bound to $\alpha > 1/p$ since in that case the Sobolev embedding $W_{\alpha,p} \hookrightarrow L^\infty$ still holds.

Another crucial point in our approach is also that quadratic nonlinearities do not map back combinations of noisy frequencies to the kernel of the linear operator $\mathcal{L}$. For example, if we change $\mathcal{L}$ slightly to $\mathcal{L} = -(4+\partial_x^2)^2$ such that $e^{\pm i2x}$ is dominant and force $e^{\pm ix}$ with a forcing term $\varepsilon\sigma\sin(x)d\beta$, then the approach presented here would fail, as new terms appear in the amplitude equation, that are much larger than order one. In order to obtain a meaningful result we need to consider smaller noise or larger distance from bifurcation. In that case it was shown in [MBK14] that on the bounded domain this leads to a constant deterministic forcing term in the amplitude equation. This is due to the fact that the quadratic nonlinearity maps the square of the noisy frequency to the dominant one, which is then averaged to a constant. We expect a similar result for the unbounded case.

7.4 Attractivity

If on the bounded domain one has a stable nonlinearity as in [BH04] e.g. the stable cubic nonlinearity $-u^3$, then the set of functions that fulfil the assumptions on the initial conditions in Theorem 7 attracts all solutions with any kind of initial values in the sense that $\|u(t)\| \leq e^{-\varepsilon^2 t}\|u(0)\| + C\varepsilon$. Since our nonlinearity does not suffice the according criteria we can only rely on the stabilising effect of the semigroup. As in [BMPS01] if we start with initial conditions $\|u(0)\|_\infty = \mathcal{O}(\varepsilon)$ then after a time logarithmic in ε the solution is of the form

$$u(x) = Ae^{ix} + \bar{A}e^{-ix} + Z_\varepsilon + R, \quad \|R\| = \mathcal{O}(\varepsilon^{2-\kappa}),$$

which means that after that time we may drop the initial value term $e^{-t(1+\partial_x^2)^2}u_s(0)$ out of equation (4.2).

A similar result is possible for the unbounded domain but in order to get control over the Fourier spectrum we need to assume $\|u(0)\|_\infty = \mathcal{O}(\varepsilon)$ and for example $\|u(0)\|_{L^2} = \mathcal{O}(\varepsilon)$. Due to the rescaling and since

$$\|\varepsilon^{-1}u(\varepsilon^{-1}x)\|_{L^2} = \varepsilon^{-1/2}\|u(x)\|_{L^2}$$

this would mean that $\|A\|_{L^2} = \varepsilon^{1/2}$ i.e. the L^2-norm of the solution to the amplitude equation would not be of order one anymore.

A Proof for the boundedness of relevant semigroups on L^∞

Here we collect the proofs for the semigroups

$$
\begin{aligned}
S_{[0,2\pi]}(t) : L^2_{per} \to L^2_{per},\, u(x) &\mapsto \sum_{k\in\mathbb{N}} e^{-t(1-k^2)^2}\hat{u}_k e^{ikx} \\
S_{\mathbb{R}}(t) : \mathcal{H}^\alpha \oplus \mathbb{R} \to \mathcal{H}^\alpha \oplus \mathbb{R},\, u(x) &\mapsto \int_{\mathbb{R}} e^{-t(1-k^2)^2}\mathcal{F}(u)(k)e^{ikx}dk \\
e^{-t\mathcal{L}^{+n}_\varepsilon} : \mathcal{H}^\alpha \oplus \mathbb{R} \to \mathcal{H}^\alpha \oplus \mathbb{R},\, u(x) &\mapsto \int_{\mathbb{R}} e^{-t(1-(\varepsilon k+n)^2)^2}\mathcal{F}(u)(k)e^{ikx}dk
\end{aligned}
$$

being bounded on $L^\infty([0,2\pi])$ and $L^\infty(\mathbb{R})$ respectively. This can be shown by applying the following Lemma which follows the ideas of Collet and Eckmann in [CE90].

Lemma A.1. *Let $\lambda(k) \in C^\infty(\mathbb{R})$ be a real valued function that fulfils*

$$\forall\, \alpha, \beta \in \mathbb{N}, t > 0 : \|k^\alpha \partial_k^\beta e^{-t\lambda(k)}\|_\infty < \infty \tag{A.1}$$

$$\sup_{t\in[0,\infty]} \|(1-\partial_k^2)e^{-t\lambda(a_t k)}\|_{L^1(\mathbb{R})} \le C, \tag{A.2}$$

where $a_t \in \mathbb{R}$ may depend on t. Then the semigroup

$$e^{tA}f := \mathcal{F}^{-1}(e^{-t\lambda(k)}(\mathcal{F}f)(k)),$$

defined on the space of tempered distributions $S'(\mathbb{R})$, is bounded on $L^\infty(\mathbb{R})$, i.e. for all $t \ge 0$:

$$\|e^{tA}f\|_\infty \le 4C\|f\|_\infty. \tag{A.3}$$

Proof. We define the Green's function of A by

$$\mathcal{G}_t := \mathcal{F}^{-1}(e^{-t\lambda(k)})$$

and because of assumption (A.1) (see for example [Str03]) we may rewrite the left-hand side of (A.3) for $t > 0$:

$$\|e^{tA}f\|_\infty = \|\mathcal{G}_t * f\|_\infty \le \|\mathcal{G}_t\|_{L^1}\|f\|_\infty.$$

So it is left to bound $\mathcal{G}_t$:

$$\begin{aligned}
\|\mathcal{G}_t\|_{L^1} &= \int_{\mathbb{R}} |\int_{\mathbb{R}} e^{ikx} e^{-t\lambda(k)} dk|\, dx \\
&= \int_{\mathbb{R}} |\int_{\mathbb{R}} e^{ikx} e^{-t\lambda(a_t k)} dk|\, dx \\
&= \int_{\mathbb{R}} (1+x^2)^{-1} |(1+x^2)\mathcal{F}^{-1}(e^{-t\lambda(a_t k)})| dx \\
&= \|(1+x^2)^{-1}\mathcal{F}^{-1}((1-\partial_k^2)e^{-t\lambda(a_t k)})\|_{L^1} \\
&\leq \|(1+x^2)^{-1}\|_{L^1} \|\mathcal{F}^{-1}((1-\partial_k^2)e^{t\lambda(a_t k)})\|_\infty \\
&\leq 4 \int_{\mathbb{R}} |(1-\partial_k^2)e^{t\lambda(a_t k)}| dk \\
&\leq 4\|(1-\partial_k^2)e^{t\lambda(a_t k)}\|_{L^1(\mathbb{R})}.
\end{aligned}$$

And now (A.3) follows from (A.2). $\square$

Remark 34. *For the operator $\partial_x^2 : \mathcal{H}^2(\mathbb{R}) \to L^2(\mathbb{R})$ we can directly calculate its Green's function which is the density of a normal distribution. As such the $L^1(\mathbb{R})$ norm of the Green's function is simply equal to one, therefore $e^{t\partial_x^2}$ is a contraction on $L^\infty(\mathbb{R})$ for all $t > 0$:*

$$\|e^{t\partial_x^2} f\|_\infty = \|\tfrac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} * f\|_\infty \leq 1 \cdot \|f\|_\infty. \tag{A.4}$$

A direct application of Lemma A.1 yields the mentioned result for the semigroups on the unbounded domain.

Corollary A.2. *The semigroups $e^{-t\mathcal{L}} := S_{\mathbb{R}}(t)$ defined in (2.2) and $e^{-t\mathcal{L}_\varepsilon^{+n}}$ defined through Definition 20 are bounded on $L^\infty(\mathbb{R})$ for all $t \geq 0$:*

$$\|e^{-t\mathcal{L}} f\|_\infty \leq C\|f\|_\infty \tag{A.5}$$

$$\|e^{-t\mathcal{L}_\varepsilon^{+n}} f\|_\infty \leq (C+n^4)\|f\|_\infty, \tag{A.6}$$

where $C > 0$ is a constant independent of t.

Proof. Let us first check condition (A.1) (i.e. that $e^{-t\lambda(k)}$ is in the Schwartz space) for $\lambda(k) := (1-k^2)^2$. For any $\alpha, \beta \in \mathbb{N}$ and $t > 0$ we have

$$k^\alpha \partial_k^\beta e^{-t\lambda(k)} = (1 + t p_{\alpha,\beta}(k)) e^{-t(1-k^2)^2} \overset{|k|\to\infty}{\to} 0,$$

where $p_{\alpha,\beta}(k)$ is a polynomial depending on α and β.

The condition (A.2) also holds as the following calculation shows. We have

$$\begin{aligned}
(1-\partial_k^2)e^{-t\lambda(a_t k)} &= (1 - t^2(\partial_k \lambda(a_t k))^2 - t\partial_k^2 \lambda(a_t k))e^{-t\lambda(a_t k)} \\
&= (1 - t^2 a_t^4(16k^2 + 32a_t^2 k^4 + 16a_t^4 k^6) - t a_t^2(-4 + 12a_t^2 k^2))e^{-t\lambda(a_t k)}.
\end{aligned}$$

For $t \geq 1$ we set $a_t = t^{-1/2} \leq 1$ to get

$$\begin{aligned}
&\int_{\mathbb{R}} |(1-\partial_k^2)e^{-t\lambda(t^{-1/2}k)}|dk \\
&\quad \leq 32 \int_{\mathbb{R}} (1+(1+t^{-1})k^2+t^{-1}k^4+t^{-2}k^6)e^{-(t^{1/2}-|k|)^2(1+t^{-1/2}|k|)^2}dk \\
&\quad \leq C \int_{\mathbb{R}} (1+k^6)e^{-t(1-t^{-1/2}|k|)^2}dk \\
&\quad = C \int_{\mathbb{R}} (1+k^6)e^{-(t^{1/2}-|k|)^2}dk \\
&\quad = 2C \int_{-t^{1/2}}^{\infty} (1+k^6)e^{-k^2}dk \\
&\quad \leq 2C \int_{\mathbb{R}} (1+k^6)e^{-k^2}dk < \infty
\end{aligned}$$

and for $0 < t < 1$ with $a_t = t^{-1/4}$ in a similar way:

$$\begin{aligned}
&\int_{\mathbb{R}} |(1-\partial_k^2)e^{-t\lambda(t^{-1/4}k)}|dk \\
&\quad \leq 32 \int_{\mathbb{R}} (1+(t+t^{1/2})k^2+t^{1/2}k^4+k^6)e^{-(t^{1/4}-|k|)^2(t^{1/4}+|k|)^2}dk \\
&\quad \leq 32 \int_{\mathbb{R}} (1+(t+t^{1/2})k^2+t^{1/2}k^4+k^6)e^{-(t^{1/4}-|k|)^2}dk \\
&\quad \leq 2C \int_{-t^{1/4}}^{\infty} (1+k^6)e^{-k^2}dk \\
&\quad \leq 2 \int_{\mathbb{R}} (1+k^6)e^{-k^2}dk < \infty.
\end{aligned}$$

Now we can apply Lemma A.1 which gives the bound (A.5). With this we obtain the second bound from

$$\begin{aligned}
\|e^{-t\mathcal{L}_\varepsilon^{+n}} f(x)\|_\infty &= \|e^{-t\mathcal{L}_\varepsilon^{+n}} f(x)e^{in\varepsilon^{-1}x}\|_\infty \\
&= \|e^{-t\mathcal{L}_\varepsilon^{+n}}(1-P_c)f(x)e^{in\varepsilon^{-1}x} + (1-n^2)^2 P_c f(x)e^{in\varepsilon^{-1}x}\|_\infty \\
&\leq \|e^{-t\mathcal{L}_\varepsilon}((1-P_c)f(x)e^{in\varepsilon^{-1}x})\|_\infty + (1-n^2)^2|P_c f| \\
&\leq \|e^{-\varepsilon^{-2}t\mathcal{L}}(f(\varepsilon x)e^{inx})\|_\infty + (1-n^2)^2\|f\|_\infty \\
&\leq C\|f(\varepsilon x)e^{inx}\|_\infty + (1-n^2)^2\|f\|_\infty = (C+n^4)\|f\|_\infty.
\end{aligned}$$

□

By the embedding of $L^\infty_{per}[0,2\pi]$ into $L^\infty(\mathbb{R})$ due to periodic extension we also get the desired result for $S_{[0,2\pi]}(t)$:

Corollary A.3. *Let $e^{-t\mathcal{L}}$ be the semigroup on $L^2_{per}[0,2\pi]$ defined by*

$$e^{-t\mathcal{L}}u(x) = \sum_{k\in\mathbb{N}} e^{-t(1-k^2)^2}(\mathcal{F}u)(k)e^{ikx}.$$

Then there exists a constant $C > 0$ such that for all $t \geq 0$:

$$\|e^{-t\mathcal{L}}u\|_\infty \leq C\|u\|_\infty.$$

Proof. We identify $L^\infty[0,2\pi]$ as a subset of $L^\infty(\mathbb{R})$ by the isomorphism

$$(Tu)(x + 2k\pi) = u(x).$$

The Fourier transform of Tu is given by

$$\mathcal{F}(Tu) = \mathcal{F}(\sum_{k\in\mathbb{N}} u_k e^{ikx}) = \sum_{k\in\mathbb{N}} u_k \delta(k - x)$$

which yields

$$\begin{aligned} e^{-t\mathcal{L}}(Tu) &:= (\mathcal{F}^{-1}e^{-t(1-k^2)}\mathcal{F}(Tu))(x) \\ &= \mathcal{F}^{-1}(e^{-t(1-k^2)}\sum_{k\in\mathbb{N}} u_k\delta_0(k - x)) \\ &= \sum_{k\in\mathbb{N}} e^{-t(1-k^2)}u_k e^{ikx} = T(e^{-t\mathcal{L}}u), \end{aligned}$$

where $\delta_0(x)$ is the Dirac distribution. Now as $\|Tu\|_\infty = \|u\|_\infty$ we just need to apply Lemma A.1:

$$\|e^{-t\mathcal{L}}u\|_\infty = \|T(e^{-t\mathcal{L}}u)\|_\infty = \|e^{-t\mathcal{L}}(Tu)\|_\infty \leq C\|Tu\|_\infty = C\|u\|_\infty.$$

$\square$

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